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BOUNDARY CONTROL PROBLEM ASSOCIATED WITH A PSEUDO-PARABOLIC EQUATION

FARRUKH N. DEKHKONOV

ABSTRACT. Previously, boundary control problems for a parabolic type equations were considered. A portion of the thin rod boundary has a temperature-controlled heater. Its mode of operation should be found so that the average temperature in some region reaches a certain value. In this paper, we consider the boundary control problem for a pseudo-parabolic equation in a right rectangle domain. The value of the solution with the control parameter is given in the boundary of the domain. Control constraints are given such that the average value of the solution in considered domain takes a given value. The auxiliary problem is solved by the method of separation of variables, and the problem under consideration is reduced to the Volterra integral equation. The existence theorem of admissible control is proved by the Laplace transform method.

1. Introduction

Consider the pseudo-parabolic equation in the domain $\Omega_T = \{(x,y) : 0 < x < a, \ \ 0 < y < b, \ \ 0 < t < T\}$:

$$\frac{\partial u}{\partial t} = \frac{\partial^2}{\partial t \partial x}\left(k(x)\frac{\partial u}{\partial x}\right) + \frac{\partial}{\partial x}\left(k(x)\frac{\partial u}{\partial x}\right)$$

$$+\frac{\partial^3 u}{\partial t \partial y^2} + \frac{\partial^2 u}{\partial y^2}, \quad (x,y,t) \in \Omega_T, \tag{1.1}$$

with boundary conditions

$$u(0,y,t) = \varphi(y)\,\mu(t), \quad u(a,y,t) = 0, \quad 0 < y < b, \tag{1.2}$$

$$u(x,0,t) = 0, \quad u(x,b,t) = 0, \quad 0 < x < a, \quad 0 < t < T, \tag{1.3}$$

and initial condition

$$u(x,y,0) = 0, \quad 0 \leq x \leq a, \quad 0 \leq y \leq b. \tag{1.4}$$

Assume that the function $k(x) \in C^2\big([0,a]\big)$ satisfies conditions

$$k(x) > 0, \quad k'(x) \leq 0, \quad 0 \leq x \leq a.$$

and the function $\varphi(y) \in W_2^2[0,b]$ satisfies conditions

$$\varphi(0) \ = \ \varphi(b) \ = \ 0, \quad \varphi_n \geq 0, \quad 0 \leq y \leq b, \tag{1.5}$$

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where

$$\varphi_n = \frac{2}{b} \int_0^b \varphi(y) \sin \frac{n\pi y}{b}\, dy. \tag{1.6}$$

Definition 1.1. If function $\mu(t) \in W_2^1(\mathbb{R}_+)$ satisfies the conditions $\mu(0) = 0$, $|\mu(t)| \leq 1$, we say that this function is an *admissible control.*

Problem H. For the given function $\theta(t)$ Problem H consists in looking for the admissible control $\mu(t)$ such that the solution $u(x, y, t)$ of the initial-boundary problem (1.1)-(1.4) exists and for all $t \geq 0$ satisfies the equation

$$\int_0^a \int_0^b u(x, y, t)\, dy\, dx = \theta(t). \tag{1.7}$$

One of the models is the theory of incompressible simple fluids with decaying memory, which can be described by equation (1) (see [1]). In [2], stability, uniqueness, and availability of solutions of some classical problems for the considered equation were studied (see also [3, 4]). Point control problems for parabolic and pseudo-parabolic equations were considered. Some problems with distributed parameters impulse control problems for systems were studied in [5, 6]. We recall that the time-optimal control problem for partial differential equations of parabolic type was first investigated in [7] and [8]. More recent results concerned with this problem were established in [9, 10, 11, 12, 13, 14, 15, 16]. Detailed information on the problems of optimal control for distributed parameter systems is given in the monographs [17, 18, 19, 20]. General numerical optimization and optimal boundary control have been studied in a great number of publications such as [21]. The practical approaches to optimal control of the heat conduction equation are described in publications like [22].

Consider the following eigenvalue problem

$$\frac{\partial}{\partial x}\left(k(x)\frac{\partial v_{m,n}(x, y)}{\partial x}\right) + \frac{\partial^2 v_{m,n}}{\partial y^2} = -\lambda_{m,n}\, v_{m,n}(x, y), \quad (x, y) \in \Omega, \tag{1.8}$$

with boundary conditions

$$v_{m,n}(x, y) \mid_{\partial\Omega} = 0, \tag{1.9}$$

where $v_{m,n}(x, y) = \vartheta_m(x)\, \omega_n(y)$ and these functions are solutions of the following eigenvalue problems

$$\frac{d}{dx}\left(k(x)\frac{d\vartheta_m(x)}{dx}\right) + \mu_m\, \vartheta_m(x) = 0, \quad 0 < x < a, \tag{1.10}$$

with boundary conditions

$$\vartheta_m(0) = \vartheta_m(a) = 0, \quad 0 \leq x \leq a, \tag{1.11}$$

and

$$\frac{d^2\omega_n(y)}{dy^2} + \nu_n\, \omega_n(y) = 0, \quad \omega_n(0) = \omega_n(b) = 0. \tag{1.12}$$

It is worth noting

$$\lambda_{m,n} = \mu_m + \nu_n, \quad \nu_n = \frac{n^2\pi^2}{b^2}, \quad \text{and} \quad \omega_n(y) = \sin\frac{n\pi}{b}.$$

It is well-know that this problem is self-adjoint in $L_2(\Omega)$ and there exists a sequence of eigenvalues $\{\lambda_{m,n}\}$ so that

$$0 < \lambda_{11} \leq ... \leq \lambda_{m,n} \to \infty, \quad m, n \to \infty.$$

The corresponding eigenfuction $v_{m,n}$ form a complete orthonormal system in $L_2(\Omega)$ and these function belong to $C(\bar{\Omega})$, where $\bar{\Omega} = \Omega \cup \partial\Omega$ (see, [23, 25]).

2. Main integral equation

Definition 2.1. By the solution of the problem (1.1)–(1.4) we understand the function $u(x, y, t)$ represented in the form

$$u(x, y, t) = \frac{a - x}{a}\,\varphi(y)\,\mu(t) - v(x, y, t), \tag{2.1}$$

where the function $v(x, y, t) \in C^{2,2,1}_{x,y,t}(\Omega_T) \cap C(\bar{\Omega}_T)$, $v_x \in C(\bar{\Omega}_T)$ is the solution to the problem:

$$v_t - \frac{\partial^2}{\partial t \partial x}\left(k(x)\,\frac{\partial v}{\partial x}\right) - \frac{\partial}{\partial x}\left(k(x)\,\frac{\partial v}{\partial x}\right) - \frac{\partial^3 v}{\partial t \partial y^2} - \frac{\partial^2 v}{\partial y^2}$$

$$= \frac{a - x}{a}\,\varphi(y)\,\mu'(t) + \frac{k'(x)}{a}\,\varphi(y)\,[\mu(t) + \mu'(t)] - \frac{a - x}{a}\,\varphi''(y)\,[\mu(t) + \mu'(t)],$$

with initial-boundary value conditions

$$v(x, y, t)\,|_{\partial\Omega} = 0, \quad v(x, y, 0) = 0.$$

Set

$$\beta_{m,n} = \big(\lambda_{m,n}\,a_{m,n} - b_{m,n} + c_{m,n}\big)\,\gamma_{m,n}, \tag{2.2}$$

where

$$a_{m,n} = \varphi_n \int\limits_0^a \frac{a - x}{a}\,\vartheta_m(x)\,dx, \tag{2.3}$$

$$b_{m,n} = \varphi_n \int\limits_0^a \frac{k'(x)}{a}\,\vartheta_m(x)\,dx, \quad c_{m,n} = -\varphi_n\,\nu_n \int\limits_0^a \frac{a - x}{a}\,\vartheta_m(x)\,dx, \tag{2.4}$$

and

$$\gamma_{m,n} = \int\limits_0^a \int\limits_0^b \vartheta_m(x)\,\omega_n(y)\,dy\,dx. \tag{2.5}$$

Consequently, we have (see, [23])

$$v(x, y, t) = \sum_{m,n=1}^{\infty} \frac{\vartheta_m(x)\,\omega_n(y)}{1 + \lambda_{m,n}}$$

$$\times \int\limits_0^t e^{-q_{m,n}(t-s)}\big(\mu'(s)\,[a_{m,n} + b_{m,n} - c_{m,n}] + \mu(s)\,[b_{m,n} - c_{m,n}]\big)\,ds, \tag{2.6}$$

where $q_{m,n} = \frac{\lambda_{m,n}}{1+\lambda_{m,n}}$.

From (2.1) and (2.6), we get the solution of the problem (1.1)–(1.4):

$$u(x,y,t) = \frac{a-x}{a}\varphi(y)\,\mu(t) - \sum_{m,n=1}^{\infty} \frac{\vartheta_m(x)\,\omega_n(y)}{1+\lambda_{m,n}}$$

$$\times \int_0^t e^{-q_{m,n}\,(t-s)}\big(\mu'(s)\,[a_{m,n}+b_{m,n}-c_{m,n}] + \mu(s)\,[b_{m,n}-c_{m,n}]\big)\,ds.$$

According to condition (1.7) and the solution of the problem (1.1)-(1.4), we may write

$$\theta(t) = \int_0^a \int_0^b u(x,y,t)\,dy\,dx = \mu(t)\int_0^a \int_0^b \varphi(y)\,\frac{a-x}{a}\,dy\,dx$$

$$- \sum_{m,n=1}^{\infty} \frac{\gamma_{m,n}}{1+\lambda_{m,n}} \int_0^t e^{-q_{m,n}(t-s)}\big(\mu'(s)\,[a_{m,n}+b_{m,n}-c_{m,n}] + \mu(s)\,[b_{m,n}-c_{m,n}]\big)\,ds$$

$$= \mu(t)\int_0^a \int_0^b \varphi(y)\,\frac{a-x}{a}\,dy\,dx - \sum_{m,n=1}^{\infty} \frac{(b_{m,n}-c_{m,n})\,\gamma_{m,n}}{1+\lambda_{m,n}} \int_0^t e^{-q_{m,n}(t-s)}\mu(s)ds$$

$$- \sum_{m,n=1}^{\infty} \frac{(a_{m,n}+b_{m,n}-c_{m,n})\,\gamma_{m,n}}{1+\lambda_{m,n}} \int_0^t e^{-q_{m,n}(t-s)}\mu'(s)ds$$

$$= \mu(t)\int_0^a \int_0^b \varphi(y)\,\frac{a-x}{a}\,dy\,dx - \mu(t)\sum_{m,n=1}^{\infty} \frac{(a_{m,n}+b_{m,n}-c_{m,n})\,\gamma_{m,n}}{1+\lambda_{m,n}}$$

$$- \sum_{m,n=1}^{\infty} \frac{(b_{m,n}-c_{m,n})\,\gamma_{m,n}}{1+\lambda_{m,n}} \int_0^t e^{-q_{m,n}(t-s)}\mu(s)ds$$

$$+ \sum_{m,n=1}^{\infty} \frac{(a_{m,n}+b_{m,n}-c_{m,n})\,\lambda_{m,n}\,\gamma_{m,n}}{(1+\lambda_{m,n})^2} \int_0^t e^{-q_{m,n}(t-s)}\,\mu(s)\,ds, \tag{2.7}$$

where $\gamma_{m,n}$ defined by (2.5).

Note that

$$\int_0^a \int_0^b \varphi(y)\,\frac{a-x}{a}\,dy\,dx = \int_0^a \int_0^b \Big(\sum_{m,n=1}^{\infty} a_{m,n}\,\vartheta_m(x)\,\omega_n(y)\Big)dy\,dx$$

$$= \sum_{m,n=1}^{\infty} a_{m,n}\,\gamma_{m,n}. \tag{2.8}$$

From (2.7) and (2.8), we get

$$\theta(t) = \mu(t) \sum_{m,n=1}^{\infty} \frac{(a_{m,n}\,\lambda_{m,n} - b_{m,n} + c_{m,n})\,\gamma_{m,n}}{1+\lambda_{m,n}}$$

$$+ \sum_{m,n=1}^{\infty} \frac{(a_{m,n}\,\lambda_{m,n} - b_{m,n} + c_{m,n})\,\gamma_{m,n}}{(1+\lambda_{m,n})^2} \int\limits_0^t e^{-q_{m,n}(t-s)}\,\mu(s)\,ds,$$

where $\lambda_{m,n} = \mu_m + \nu_n$.

Set

$$K(t) = \sum_{m,n=1}^{\infty} \rho_{m,n}\, e^{-q_{m,n}t}, \quad t > 0, \tag{2.9}$$

and

$$\alpha = \sum_{m,n=1}^{\infty} \frac{\beta_{m,n}}{1+\lambda_{m,n}}, \quad \rho_{m,n} = \frac{\beta_{m,n}}{(1+\lambda_{m,n})^2},$$

where $\beta_{m,n}$ defined by (2.2).

Later we prove $\beta_{m,n} \geq 0$ in proposition 3.2. Hence, $\alpha > 0$ and bounded.

Then we have the main integral equation

$$\alpha\,\mu(t) + \int\limits_0^t K(t-s)\,\mu(s)\,ds = \theta(t), \quad t > 0. \tag{2.10}$$

3. Main Result

Denote by $W(M)$ the set of function $\theta \in W_2^2(-\infty,+\infty)$, $\theta(t) = 0$ for $t \leq 0$ which satisfies the condition

$$\|\theta\|_{W_2^2(R_+)} \leq M.$$

Theorem 3.1. *There exists $M > 0$ such that for any function $\theta \in W(M)$ the solution $\mu(t)$ of the equation (2.10) exists and satisfies condition*

$$|\mu(t)| \;\leq\; 1.$$

Proposition 3.2. *For the cofficients $\beta_{m,n}$ defined by* (2.2) *the estimate*

$$0 \leq \beta_{m,n} \leq C\,\frac{\varphi_n}{n}, \quad m,n = 1,2,... \tag{3.1}$$

is valid, where φ_n defined by (1.6).

Proof. **Step 1.** According to (1.10), (1.11) and (2.3), we first calculate the following integral

$$a_{m,n}\,\mu_m = \mu_m\varphi_n \int\limits_0^a \frac{a-x}{a}\,\vartheta_m(x)\,dx$$

$$= -\varphi_n \int\limits_0^a \frac{a-x}{a}\,\frac{d}{dx}\left(k(x)\frac{d\vartheta_m(x)}{dx}\right)dx = \varphi_n\,k(0)\,\vartheta_m'(0)$$

$$-\varphi_n \int\limits_0^a \frac{k(x)}{a}\,\vartheta_m'(x)dx = \varphi_n\, k(0)\,\vartheta_m'(0) + \varphi_n \int\limits_0^a \frac{k'(x)}{a}\,\vartheta_m(x)dx$$

$$= \varphi_n\, k(0)\,\vartheta_m'(0) + b_{m,n}.$$

Secondly, from (1.12) and (2.3)

$$a_{m,n}\,\nu_n = \nu_n\,\varphi_n \int\limits_0^a \frac{a-x}{a}\,\vartheta_m(x)\,dx = -c_{m,n}.$$

Then we have

$$a_{m,n}(\mu_m+\nu_n) - b_{m,n} + c_{m,n} = \varphi_n\, k(0)\,\vartheta_m'(0). \tag{3.2}$$

Step 2. Now we integrate the Eq. (1.10) from 0 to x

$$k(x)\vartheta_m'(x) - k(0)\vartheta_m'(0) = -\mu_m \int\limits_0^x \vartheta_m(\tau)d\tau,$$

and according to $k(x)>0,\ x\in[0,a]$, we can write

$$\vartheta_m'(x) - \frac{1}{k(x)}k(0)\vartheta_m'(0) = -\frac{\mu_m}{k(x)} \int\limits_0^x \vartheta_m(\tau)d\tau. \tag{3.3}$$

Thus, we integrate the Eq. (3.3) from 0 to a. Then we have

$$\vartheta_m(a) - \vartheta_m(0) - k(0)\vartheta_m'(0)\int\limits_0^a \frac{dx}{k(x)} = -\mu_m \int\limits_0^a \frac{1}{k(x)}\Big(\int\limits_0^x \vartheta_m(\tau)d\tau\Big)dx. \tag{3.4}$$

From (1.11) and (3.4), we get

$$k(0)\vartheta_m'(0)\int\limits_0^a \frac{dx}{k(x)} = \mu_m \int\limits_0^a \frac{1}{k(x)}\Big(\int\limits_0^x \vartheta_m(\tau)d\tau\Big)dx.$$

Then

$$k(0)\vartheta_m'(0) = \mu_m \int\limits_0^a G(\tau)\vartheta_m(\tau)d\tau, \tag{3.5}$$

where

$$G(\tau) = \int\limits_\tau^a \frac{dx}{k(x)}\Big(\int\limits_0^a \frac{dx}{k(x)}\Big)^{-1}.$$

According to $G(\tau)>0$ and from (3.5), we have (see, [25])

$$\vartheta_m'(0)\int\limits_0^a \vartheta_m(\tau)d\tau \geq 0. \tag{3.6}$$

Consequently, according to (1.5), (3.2) and (3.6), we get the following estimate

$$\beta_{m,n} = \left(a_{m,n}\,(\mu_m+\nu_n) - b_{m,n} + c_{m,n}\right)\gamma_{m,n}$$

$$= \varphi_n \, k(0)\vartheta'_m(0) \int_0^a \vartheta_m(x)dx \int_0^b \omega_n(y)dy$$

$$= \frac{b\,[1-(-1)^n]}{n\pi} \varphi_n \, k(0)\vartheta'_m(0) \int_0^a \vartheta_m(x)dx \geq 0.$$

Step 3. It is clear that if $k(x) \in C^1([0,a])$, we may write the estimate (see, [25, 26])

$$\max_{0 \leq x \leq a} |\vartheta'_m(x)| \leq C\mu_m^{1/2}.$$

Therefore,

$$|\vartheta'_m(0)| \leq C\mu_m^{1/2}, \quad |\vartheta'_m(a)| \leq C\mu_m^{1/2},$$

From (1.10), we can write

$$k(a)\vartheta'_m(a) - k(0)\vartheta'_m(0) = -\mu_m \int_0^a \vartheta_m(x)dx.$$

Then we obtain

$$|\gamma_{m,n}| = \frac{b\,[1-(-1)^n]}{n\pi} \left| \int_0^a \vartheta_m(x)dx \right|$$

$$= \frac{b\,[1-(-1)^n]}{n\pi} \left| \frac{k(a)\vartheta'_m(a) - k(0)\vartheta'_m(0)}{\mu_m} \right| \leq C \, \frac{\mu_m^{-1/2}}{n}.$$

Consequently,

$$|\beta_{m,n}| = \varphi_n \, k(0)|\vartheta'_m(0) \, \gamma_{m,n}| \leq C \, \frac{\varphi_n}{n}.$$

□

Proposition 3.3. *A function $K(t)$ defined by* (2.9) *is continuous on the half-line $t \geq 0$.*

Proof. Indeed, according to proposition 3.2, we can write

$$0 \leq \rho_{m,n} = \frac{\beta_{m,n}}{(1+\lambda_{m,n})^2} \leq C \, \frac{\varphi_n}{n\,(1+\lambda_{m,n})^2}.$$

It is clear that, from (3.1) function $K(t)$ is positive. It is known from the general theory that if $k(x)$ is a smooth function, the following estimate is valid (see, [26]):

$$\mu_m = \frac{m^2\pi^2}{p^2} + O(m^{-2}), \quad p = \int_0^a \frac{dx}{\sqrt{k(x)}},$$

where μ_m are the eigenvalues of problem (1.10)-(1.11).

Then we have

$$0 < K(t) \leq \text{const} \sum_{m,n=1}^{\infty} \frac{\varphi_n}{n\,(1+\lambda_{m,n})^2}.$$

where $\lambda_{m,n} = \mu_m + \nu_n$. □

We write integral equation (2.10)

$$\alpha\,\mu(t) + \int\limits_0^t K(t-s)\mu(s)ds = \theta(t), \quad t > 0.$$

For solve equation (2.10), we use the Laplace transform method. We introduce the notation

$$\widetilde{\mu}(p) = \int\limits_0^\infty e^{-pt}\,\mu(t)\,dt.$$

Then we use Laplace transform obtain the following equation

$$\widetilde{\theta}(p) = \int\limits_0^\infty e^{-pt}\,dt \int\limits_0^t K(t-s)\mu(s)ds + \alpha \int\limits_0^\infty e^{-pt}\,\mu(t)\,dt = \widetilde{K}(p)\,\widetilde{\mu}(p) + \alpha\,\widetilde{\mu}(p).$$

Consequently, we get

$$\widetilde{\mu}(p) = \frac{\widetilde{\theta}(p)}{\alpha + \widetilde{K}(p)}, \quad \text{where } \; p = a + i\xi, \quad a > 0,$$

and

$$\mu(t) = \frac{1}{2\pi i} \int\limits_{a-i\xi}^{a+i\xi} \frac{\widetilde{\theta}(p)}{\alpha + \widetilde{K}(p)}\, e^{pt} dp = \frac{1}{2\pi} \int\limits_{-\infty}^{+\infty} \frac{\widetilde{\theta}(a+i\xi)}{\alpha + \widetilde{K}(a+i\xi)}\, e^{(a+i\xi)t} d\xi. \tag{3.7}$$

Then we can write

$$\widetilde{K}(p) = \int\limits_0^\infty K(t)e^{-pt}\,dt = \sum_{m,n=1}^\infty \rho_{m,n} \int\limits_0^\infty e^{-(p+q_{m,n})t}\,dt = \sum_{m,n=1}^\infty \frac{\rho_{m,n}}{p + q_{m,n}},$$

where $K(t)$ defined by (2.9) and

$$\alpha + \widetilde{K}(a+i\xi) = \alpha + \sum_{m,n=1}^\infty \frac{\rho_{m,n}}{a + q_{m,n} + i\xi} = \alpha + \sum_{m,n=1}^\infty \frac{\rho_{m,n}\,(a + q_{m,n})}{(a + q_{m,n})^2 + \xi^2}$$

$$-i\xi \sum_{m,n=1}^\infty \frac{\rho_{m,n}}{(a + q_{m,n})^2 + \xi^2} = \mathrm{Re}(\alpha + \widetilde{K}(a+i\xi)) + i\,\mathrm{Im}(\alpha + \widetilde{K}(a+i\xi)),$$

where

$$\mathrm{Re}(\alpha + \widetilde{K}(a+i\xi)) = \alpha + \sum_{m,n=1}^\infty \frac{\rho_{m,n}\,(a + q_{m,n})}{(a + q_{m,n})^2 + \xi^2},$$

$$\mathrm{Im}(\alpha + \widetilde{K}(a+i\xi)) = -\xi \sum_{m,n=1}^\infty \frac{\rho_{m,n}}{(a + q_{m,n})^2 + \xi^2}.$$

We know that

$$(a + q_{m,n})^2 + \xi^2 \le [(a + q_{m,n})^2 + 1](1 + \xi^2),$$

and we have the following inequality

$$\frac{1}{(a+q_{m,n})^2+\xi^2} \geq \frac{1}{1+\xi^2}\,\frac{1}{(a+q_{m,n})^2+1}. \tag{3.8}$$

Consequently, according to estimate (3.1) and (3.8) we have the following estimates

$$\begin{aligned}|\mathrm{Re}(\alpha+\widetilde{K}(a+i\xi))| &= \alpha+\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}\,(a+q_{m,n})}{(a+q_{m,n})^2+\xi^2}\\ &\geq \frac{1}{1+\xi^2}\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}\,(a+q_{m,n})}{(a+q_{m,n})^2+1}=\frac{C_{1a}}{1+\xi^2},\end{aligned} \tag{3.9}$$

and

$$\begin{aligned}|\mathrm{Im}(\alpha+\widetilde{K}(a+i\xi))| &= |\xi|\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}}{(a+q_{m,n})^2+\xi^2}\\ &\geq \frac{|\xi|}{1+\xi^2}\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}}{(a+q_{m,n})^2+1}=\frac{C_{2a}\,|\xi|}{1+\xi^2},\end{aligned} \tag{3.10}$$

where C_{1a}, C_{2a} as follows

$$C_{1a}=\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}\,(a+q_{m,n})}{(a+q_{m,n})^2+1},\quad C_{2a}=\sum_{m,n=1}^{\infty}\frac{\rho_{m,n}}{(a+q_{m,n})^2+1}.$$

From (3.9) and (3.10), we have the following estimate

$$|\alpha+\widetilde{K}(a+i\xi)|^2=|\mathrm{Re}(\alpha+\widetilde{K}(a+i\xi))|^2+|\mathrm{Im}(\alpha+\widetilde{K}(a+i\xi))|^2\geq\frac{\min(C_{1a}^2,C_{2a}^2)}{1+\xi^2},$$

and

$$|\alpha+\widetilde{K}(a+i\xi)|\geq\frac{C_a}{\sqrt{1+\xi^2}},\quad \text{where}\quad C_a=\min(C_{1a},C_{2a}). \tag{3.11}$$

Then, when $a\to 0$ from (3.7), we obtain

$$\mu(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty}\frac{\widetilde{\theta}(i\xi)}{\alpha+\widetilde{K}(i\xi)}\,e^{i\xi t}d\xi. \tag{3.12}$$

Lemma 3.4. *Let $\theta(t)\in W(M)$. Then for the image of the function $\theta(t)$ the following inequality*

$$\int_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi\leq C\,\|\theta\|_{W_2^2(R_+)},$$

is valid.

Proof. We calculate the Laplace transform of a function $\theta(t)$ as follows

$$\widetilde{\theta}(a+i\xi)=\int_0^{\infty}e^{-(a+i\xi)t}\theta(t)\,dt=-\theta(t)\,\frac{e^{-(a+i\xi)t}}{a+i\xi}\Big|_{t=0}^{t=\infty}+\frac{1}{a+i\xi}\int_0^{\infty}e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

then, we get

$$(a+i\xi)\,\widetilde{\theta}(a+i\xi)=\int\limits_{0}^{\infty}e^{-(a+i\xi)t}\,\theta'(t)\,dt,$$

and for $a\to 0$ we have

$$i\xi\,\widetilde{\theta}(i\xi)=\int\limits_{0}^{\infty}e^{-i\xi t}\,\theta'(t)\,dt.$$

Also, we can write the following equality

$$(i\xi)^2\,\widetilde{\theta}(i\xi)=\int\limits_{0}^{\infty}e^{-i\xi t}\,\theta''(t)\,dt.$$

Then we have

$$\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2d\xi\leq C_1\,\|\theta\|^2_{W_2^2(R_+)}. \tag{3.13}$$

Consequently, according to (3.13) we get the following estimate

$$\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi=\int\limits_{-\infty}^{+\infty}\frac{|\widetilde{\theta}(i\xi)|(1+\xi^2)}{\sqrt{1+\xi^2}}$$

$$\leq\Bigg(\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|^2(1+\xi^2)^2d\xi\Bigg)^{1/2}\Bigg(\int\limits_{-\infty}^{+\infty}\frac{1}{1+\xi^2}d\xi\Bigg)^{1/2}\leq C\,\|\theta\|_{W_2^2(R_+)}.$$

□

Proof of the Theorem 3.1 We prove that $\mu\in W_2^1(\mathbb{R}_+)$. Indeed, according to (3.11) and (3.12), we obtain

$$\int\limits_{-\infty}^{+\infty}|\widetilde{\mu}(\xi)|^2(1+|\xi|^2)\,d\xi\;=\;\int\limits_{-\infty}^{+\infty}\left|\frac{\widetilde{\theta}(i\xi)}{\alpha+\widetilde{K}(i\xi)}\right|^2(1+|\xi|^2)\,d\xi$$

$$\leq C\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|^2(1+|\xi|^2)^2\,d\xi\;=\;C\|\theta\|^2_{W_2^2(\mathbb{R})}.$$

Further,

$$|\mu(t)-\mu(s)|\;=\;\left|\int\limits_{s}^{t}\mu'(\tau)\,d\tau\right|\;\leq\;\|\mu'\|_{L_2}\sqrt{t-s}.$$

Hence, $\mu\in\operatorname{Lip}\alpha$, where $\alpha=1/2$. Then, from (3.11), (3.12) and lemma 3.4, we can write

$$|\mu(t)|\leq\frac{1}{2\pi}\int\limits_{-\infty}^{+\infty}\frac{|\widetilde{\theta}(i\xi)|}{|\alpha+\widetilde{K}(i\xi)|}d\xi\leq\frac{1}{2\pi C_0}\int\limits_{-\infty}^{+\infty}|\widetilde{\theta}(i\xi)|\sqrt{1+\xi^2}d\xi$$

$$\leq \frac{C}{2\pi C_0}\|\theta\|_{W_2^2(R_+)} \leq \frac{C\,M}{2\pi C_0} = 1.$$

As M we took

$$M = \frac{2\pi C_0}{C}.$$

Theorem 3.1 is proved.

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Farrukh N. Dekhkonov: Department of Mathematics, Namangan State University, Namangan, Uzbekistan

E-mail address: f.n.dehqonov@mail.ru