Stochastic Modelling and Computational Sciences
Vol. 3 No. 1 (June, 2023)

Received: 19th March 2023 *Revised: 24th May 2023* *Accepted: 30th June 2023*

ABOUT THE NEW WAY FOR SOLVING SOME PHYSICAL PROBLEMS DESCRIBED BY ODE OF THE SECOND ORDER WITH THE SPECIAL STRUCTURE

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Abstract. Mathematical models for many theoretical and practical problems of natural sciences have been described by the ODEs. Such processes are common in the investigation of physical processes, which the scientists have begun to study from Newton (it is enough to remember Newton's laws). In recent times it often arises the necessity to construct mathematical models for some physical processes by using ODEs of the second order with the special structure (as the Schturm-Liouville, Scrhödinger and others problems). As is known there are wide classes of methods for solving the initial-value problem for the ODE of the second order with the special structures. One of the classical efficiency methods for solving these problems is the Störmer-Verlet method. Here, by developing this method have constructed the new class of hybrid methods with the constant coefficients.

1. Introduction

By taking into account that many theoretical and applied tasks are reduced to solve the initial value problem for ODE of the second order, here have considered to investigation of following problem:

$$y''(x) = F(x, y, y'),\ \ y'(x_0) = y'_0,\ \ y(x_0) = y_0,\ \ x_0 <= x <= X. \qquad (1.1)$$

There are wide classes of numerical methods for solving this problem. Here for solving problem (1.1) proposed to use the multistep multiderivative methods with constant coefficients having the hybrid types. Let us assume that the problem (1.1) has a unique solution $y(x)$, which is defined in the segment $[x_0, X]$. The totality of arguments function $F(x, y, z)$ is continuous and has been defined in some close domain.

Many famous scientists have studied finding the solution of the equation that participated in the problem (1). For the illustration of this, let us to consider the following generalization of the known Scrhödinger and Sturm-Liouville equations, which can be presented as: $y'' = \phi(x, y)$ (see [1, p.150-152], [2, p.111-113], [3, p.277]). Note that the Scrhödinger and Sturm-Liouville problems are usually

2000 *Mathematics Subject Classification.* Primary 45D05; Secondary 34A12.

Key words and phrases. Initial value problem for ODE with special structure, Störmer-Verlet method, hybrid method of Störmer type, multistep second derivative method, finite-difference equation, stability and degree.

* This work was supported by the Science Development Foundation under the President of Republic of Azerbaijan - Grant AEF-MCG-2022-1(42)-12/4/1-M-4 (for Vagif Ibrahimov). 2020-2025.

formulated by the boundary-value problem for the above mentioned equation (see for example [1, 2, 3]), which can be reduced two initial value problem for the ODE of the second order (see for example [4, 5, 6]).

As was noted above, the problem (1.1) has been investigated by many authors using one-step or multistep methods (see for example [7, 8, 9]). Here, for solving the problem (1.1) proposed to use some generalization of the Störmer-Verlet methods, for the construction of which has used the hybrid method with the special structure. To define the maximum value of the order of accuracy for proposed methods, I have used the method of unknown coefficients and the theory of nonlinear systems of algebraic equations (see for example [10, 11, 12, 13, 14, 15]).

Here the construction of numerical methods have used different schemes. Noting that for solving a specific physical problem, Störmer has constructed one simple method, which has been generalized by some scientists and in fundamental form has been explored by Dahlquist (see [7]). Usually the Störmer-Verlet method has been applied to solve the initial-value problem for the above-mentioned ODE of the second order with special structure. For the determination of the advantages and disadvantages of the Störmer-Verlet method, let us consider the construction of the Störmer-Verlet method by using some simple problems. For this aim, suppose that the right hand side of the ODE in the problem (1.1), can be presented as : $F(x, y, y') = f(x)$. In this case, from the problem (1.1) one can be receive the following:

$$y'' = f(x),\ y'(x_0) = y_0',\ y(x_0) = y_0,\ x \in [x_0, X]. \tag{1.2}$$

It is easy to prove that the solution of this problem can be written as:

$$y(x) = y(x_0) + y'(x_0)(x - x_0) + \int_{x_0}^{x} (x - s) f(s) ds,$$

$$y'(x) = y'(x_0) + \int_{x_0}^{x} f(s) ds,$$

$$2y'(x_0) = 2y'(x) - 2\int_{x_0}^{x} f(s).$$

From here receive that:

$$y(x+h) = y(x_0) + y'(x_0)(x - x_0) + hy'(x_0) + \int_{x_0}^{x+h} (x - s) f(s) ds + h\int_{x_0}^{x+h} f(s) ds,$$

$$y(x-h) = y(x_0) + y'(x_0)(x - x_0) - hy'(x_0) + \int_{x_0}^{x-h} (x - s) f(s) ds - h\int_{x_0}^{x-h} f(s).$$

Here, the above used presentation of the solution of problem (1.1), has been chosen so that the construction Störmer-Verlet method should have been done simply.

It's not hard to show that one can be receive the following from the proposed above:

$$y(x) = y(x-h) + hy'(x-h) + \int_{x-h}^{x} (x-s)f(s)ds,$$

$$y(x+h) = y(x-h) + 2hy'(x-h) + \int_{x-h}^{x+h} (x-s)f(s)ds + h\int_{x-h}^{x+h} f(s)ds. \tag{1.3}$$

It's obvious that there are many such representations. For example the following:

$$y(x+h) = y(x-h) + h(y'(x+h) + y'(x-h)) + \int_{x-h}^{x+h} (x-s)f(s)ds.$$

By the application of some quadrature method to calculation of the definite integrals participated above presented equalities and compares of these equalities, receive that the numerical methods for solving of the problem (1.2) in one version can be written as (integral participated in (1.3) can be take as the double integral):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = h\sum_{i=0}^{k} \beta_i y'_{n+i} + h^2 \sum_{i=0}^{k} \gamma_i f_{n+i}. \tag{1.4}$$

Here k-is the order of the finite-difference equation of (1.4). This method has been investigated by many authors (see for example [7, 8, 9, 10, 11]).

To verify this presentation let us use the equations of (1.3) to put $x = x_n$. In this case receive:

$$\begin{aligned} y_n &= y_{n-1} + hy'_{n-1} + \int_{x_{n-1}}^{x_n} (x_n - s)f(s)ds, \\ y_{n+1} &= y_{n-1} + 2hy'_{n-1} + \int_{x_n-h}^{x_n+h} (x-s)f(s)ds + h\int_{x_n-h}^{x_n+h} f(s)ds. \end{aligned} \tag{1.5}$$

If you use some quadrature formula to computation of definite integrals participated in the system of (1.5), then after generalization of which one can be received the method of (1.4). Note that in the method (1.4) participated the values of the function of $y'(x)$.

For using method (1.4) it is necessary to find the values y'_m $(m = 0, 1, 2, ...)$. For this aim, let us consider the following:

$$y'(x+h) = y'(x) + \int_{x}^{x+h} f(s)ds; \; y'(x+h) = y'(x-h) + \int_{x-h}^{x+h} f(s)ds.$$

2. Construction of a Multistep Methods With the Special Structure

By using these and some other equality for the calculation of the value y'_m one can be recommended to use the following (by generalize many known quadrature formulas):

$$\sum_{i=0}^{k} \alpha'_i y'_{n+i} = h\sum_{i=0}^{k} \beta'_i y''_{n+i}. \tag{2.1}$$

This method can be derived from the known multistep method, which is applied to solve ODE of the first order by the change of $y(x)$ with the $y'(x)$. Thus we meet with a task in which the function $F(x, y, y')$ independent from the $y'(x)$, it is to say that $F(x, y, y') = \phi(x, y)$ performed. In this case, it is not necessary to use the method (2.1). For the shown this phenomena let us consider the following equalities:

$$y(x+h) - y(x-h) = 2hy'(x-h) + \int_{x-h}^{x+h} (x-s)f(s)ds + h\int_{x-h}^{x+h} f(s)ds,$$

$$y(x+h) - y(x) = 2hy'(x-h) + \int_{x}^{x+h} (x-s)f(s)ds + h\int_{x-h}^{x+h} f(s)ds.$$

From these equalities it follows the next:

$$y(x+h)-2y(x)+y(x-h) = \int_{x}^{x+h} (x-s)f(s)ds - \int_{x-h}^{x} (x-s)f(s)ds + h\int_{x-h}^{x+h} f(s)ds. \tag{2.2}$$

Let us to put $x = x_n + (k-1)h$ in the equality (2.2). Then receive:

$$y_{n+k} - 2y_{n+k-1} + y_{n+k-2} = \int_{x_{n+k-1}}^{x_{n+k}} (x_{n+k-1} - s)f(s)ds - \tag{2.3}$$

$$-\int_{x_{n+k-2}}^{x_{n+k-1}} (x_{n+k-1} - s)f(s)ds + h\int_{x_{n+k-2}}^{x_{n+k}} f(s)ds.$$

And now let us consider the construction of a multistep method with a special structure. For this aim, if here proposed to applied of some quadrature formula for the calculation of definite integrals participated in equation (2.3), then receive

$$y_{n+k} - 2y_{n+k-1} + y_{n+k-2} = h^2 \sum_{i=0}^{k} \gamma_i f_{n+i}, \tag{2.4}$$

here the coefficients of $\gamma_i (i = 0, 1, ..., k)$ are calculated by using the values of the coefficients of quadrature formula which has been applied to calculation of the definite integrals participated in the equality of (2.3).

After generalization of the linear part of the method (2.4), receive the following multistep second derivative method with constant coefficients:

$$\sum_{i=0}^{k} \overline{\alpha} y_{n+i} = h^2 \sum_{i=0}^{k} \overline{\beta_i} f_{n+i}. \tag{2.5}$$

By the comparison of the methods (1.4) and (2.5) received that method (2.5) can be obtained from the method (1.4) as the partial case. It is not difficult to show that these methods have different properties. As is known one of the basic conceptions for comparison of numerical methods is the degree and the other is its stability, which can be defined by the following way (see for example [7, 8, 9, 10]).

Definition 2.1. The integer values p is called as the degree for the method of (1.4) if the following asymptotic equality takes place:

$$\sum_{i=0}^{k}(\alpha_i y(x+ih) - h\beta_i y'(x+ih) - h^2\gamma_i y''(x+ih)) = O(h^{p+1}), h \to 0. \quad (2.6)$$

Definition 2.2. Method of (1.4) is called as stable if the roots of the polynomial $\rho(\lambda) = \alpha_k\lambda_k + \alpha_{k-1}\lambda_{k-1} + ... + \alpha_1\lambda + \alpha_0$ lie in the unit circle on the boundary of which there is no multiply roots.

By the investigation of the method (2.4), receive that the method (2.4) can not be stable (see definition 2), because the polynomial $\rho(\lambda)$ for the method (2.4) has the following form:

$$\rho(\lambda) = \lambda^2 - 2\lambda + 1.$$

How it follows from here, that characteristic polynomial for method (2.4) has the multiple root on the boundary of the unique circle ($\lambda = 1$ double root). It follows from here that the method (2.4) can not be stable. But it is not difficult to verify that the exact solution of the problem (1.1) satisfies the equality of (2.3). Taking into account this phenomena, some authors have used the following definition (see for example [7, 8, 9, 10]):

Definition 2.3. Method (2.5) is called as the stable if the roots of the polynomial $\overline{\rho}(\lambda) = \overline{\alpha}_k\lambda^k + \overline{\alpha}_{k-1}\lambda^{k-1} + ... + \overline{\alpha}_1\lambda + \overline{\alpha}_0$ lie in the unit circle on the boundary of which there is no multiple roots, without double root $\lambda = 1$.

Definition 2.4. The integer values p is called as the degree for the method of (10) if the following asymptotic equality holds:

$$\sum_{i=0}^{k}(\overline{\alpha_i} y(x+ih)) - h^2\overline{\beta_i} y''(x+ih)) = O(h^{p+2}), h \to 0. \quad (2.7)$$

From here, receive that the method (2.5) is an independent object for the investigation. Note that method (2.5) was investigated by many authors (see for example [4, 5, 6, 7, 8, 9, 10, 11, 12, 13]). And have defined the conditions which must satisfy the coefficients of method (2.5) for its convergence. The aim of this work is the definition of some relation between the exact and numerical solution of the problem (1.1). As is known, usually, for finding numerical solutions of the problem (1.1) have been used methods, which are called as multistep method with constant coefficients or the finite difference methods. For construction more exact methods here have proposed to use the forward-jumping (advanced) and hybrid methods, so these methods have some advantages. For the construction methods of type (2.5), let us use the formula which has used in the presentation of exact solution of the following problem:

$$y''(x) = \phi(x, y(x)), y'(x_0) = y'_0, y(x_0) = y_0. \quad (2.8)$$

For this aim, consider approximation of definite integrals participated in the equality of (2.2). As is known there are class methods for the calculation of definite integrals. Here proposed to use the following method, which can compares with the equality of (2.4):

$$\int_{x_{n+1}}^{x_{n+k}}(x_{n+1}-s)f(s)ds-\int_{x_n}^{x_{n+k-1}}(x_{n+1}-s)f(s)ds+h\int_{x_n}^{x_{n+k}}f(s)ds=$$

$$h^2\sum_{i=0}^{k}\overline{\beta_i}f_{n+i}+h^2\sum_{i=0}^{k}\overline{\gamma i}f_{n+i+\nu_i},\quad (|\nu_i|<1, i=0,1,...,k). \tag{2.9}$$

It is easy to understand that for the calculation one of the definite integral participants in (14), has used Gauss or Chebishev method.

Note that for the calculation of definite integrals one can used the method described in [12].

In the case $\nu_i = 0(i = 0, 1, ..., k)$ it follows from the formula (2.9) the known quadrature methods. In other cases, the method (14) receives the new methods, the properties of which are depending on the values of coefficients of the formula (2.9) and from the values of $\nu_i(i = 0, 1, ..., k)$. It is evident that depending on the method used for finding values of the coefficients of the formula (2.9), that can be written in different forms.

Thus, we described one way to determine the value of the coefficients of the multistep second derivative methods. However, the sequence of operations for calculating the values of coefficients is very complicated. Therefore, we will try to describe one way to determine the values of the coefficients of the method (2.5) or (2.9).

For this aim, let us applied the method of unknown coefficients to determine the values of the coefficients which are participated in the formula (2.5), then receive methods which can usually be called as the finite-difference method. To determine the values of coefficients, receive the nonlinear system of algebraic equations. For construction of the named system, let us to use the following Taylor series:

$$y(x+ih)=y(x)+ihy'(x)+\frac{(ih)^2}{2!}y''(x)+...+\frac{(ih)^{p+1}}{(p+1)!}y^{p+1}(x)+O(h^{p+2}),$$

$$y''(x+ih)=y''(x)+ihy'''(x)+\frac{(ih)^2}{2!}y^{IV}(x)+...+\frac{(ih)^{p-1}}{(p-1)!}y^{p-1}(x)+O(h^p).$$

By using these equalities in the asymptotic equality of (2.7) one can receive:

$$\sum_{i=0}^{k}\overline{\alpha_i}y(x)+h\sum_{i=0}^{k}i\overline{\alpha_i}y'(x)+h^2\sum_{i=0}^{k}(\tfrac{i^2}{2!}\overline{\alpha_i}-\overline{\beta_i})y''(x)+...$$

$$+h^{p+1}\sum_{i=0}^{k}(\tfrac{i^{p+1}}{(p+1)!}\overline{\alpha_i}-\tfrac{i^{p-1}}{(p-1)!}\overline{\beta i})y^{p+1}(x)+O(h^{p+2})=O(h^{p+2}),h\to 0. \tag{2.10}$$

It follows from here that if the method of (2.5) has the degree of p, then by the comparison of the asymptotic equalities (2.7) and (2.10) receive that the following must satisfy (see for example [16, 17, 18, 19, 20, 21]):

$$\begin{gathered}\sum_{i=0}^{k}(\overline{\alpha_i}y(x)+ih\overline{\alpha_i}y'(x)+h^2(\tfrac{i^2}{2!}\overline{\alpha_i}-\overline{\beta_i})y''(x)+...\\+h^{p+1}(\tfrac{i^{p+1}}{(p+1)!}\overline{\alpha_i}-\tfrac{i^{p-1}}{(p-1)!}\overline{\beta_i})y^{(p+1)}(x))=0.\end{gathered} \tag{2.11}$$

By taking into account that the systems $1, x, x^2, ..., x^{p+1}$ or $y(x), y'(x$ $y^{(p+1)}(x)$ $(y^j(x) \neq 0, j = 0, 1, ..., p+1)$ is independent, receive that for satisfying the equality of (2.11) the following system must be has the solution (see for example, [13], [18, 19, 20, 21, 22, 23, 24, 25]):

$$\sum_{i=0}^{k} \overline{\alpha_i} = 0; \sum_{i=0}^{k} \overline{i\alpha_i} = 0; \sum_{i=0}^{k} (\frac{i^j}{j!}\overline{\alpha_i} - \frac{i^{j-2}}{(j-2)!}\overline{\beta_i}) = 0, j = 2, 3, ..., p+1. \quad (2.12)$$

Thus, for finding the coefficients of the method (2.5) receive the linear system of algebraic equations. It is clear that the error for the method (2.5) can be estimated by the error of the quadrature methods. For the sake of objectivity let us note that at the application of the method (2.5) to solving initial-value problems some errors arise.

If method (2.5) is stable, then all the errors that arise in using method (2.5) are bounded. Therefore, let us investigate the system of (2.12). In this system the amount of the unknowns equal to $2k+2$, but the amount of the equations equals $p+2$. It is not difficult to prove that the linear system has the unique solution for the case $p = 2k$. But this equality for the stable methods of type (2.5) can be written as: $p \leq 2[k/2] + 2$. And also the constant k must satisfy the condition $k \geq 2$. This condition follows from the equality (2.4).

Thus proved the next lemma:

Lemma 2.5. *If method* (2.5) *has the degree of p, then satisfies its coefficients the system of* (2.12) *is necessary and sufficient. If the coefficient of method* (2.5) *satisfies the condition* (2.12)*, then the method* (2.5) *will have the degree p, which satisfies the condition $p \leq [k/2] + 2$ for the stable and the condition $p \leq 2k$ for other methods.*

These boundaries have been obtained by various authors (see for example [7, 5, 8, 11, 12, 14, 16, 17]).

And now let us to consider investigation of the following method:

$$\sum_{i=0}^{k} \overline{\alpha_i} y_{n+i} = h^2 \sum_{i=0}^{k} \overline{\beta_i} f_{n+i} + h^2 \sum_{i=0}^{k} \overline{\gamma_i} f_{n+i+\nu_i} (|\nu_i| < 1, i = 0, 1, ..., k). \quad (2.13)$$

For investigation method (2.13) let us describe the way for finding the values of the coefficients α_i, β_i,γ_i,ν_i $(i = 0, 1, 2, ..., k)$. For this aim, one can use the above presented Taylor series with the following:

$$y''(x + l_i h) = y''(x) + l_i h y'''(x) + \frac{l_i^2}{2!} h^2 y^{IV}(x) + ... + \frac{(l_i h)^{p+1}}{(p-1)!} y^{p+1}(x) + O(h^{p+2}),$$

here, $l_i = i + \nu_i$ $(i = 0, 1, ..., k)$.

By repeating the above using description for finding of the coefficients α_i, β_i,γ_i,ν_i $(i = 0, 1, 2, ..., k)$ receive the following nonlinear system of algebraic equations:

$$\sum_{i=0}^{k} \alpha_i = 0; \ \sum_{i=0}^{k} i\alpha_i = 0; \sum_{i=0}^{k} (\frac{i^j}{j!}\alpha_i - \frac{i^{j-2}}{(j-2)!}\beta_i - \frac{l_i^{j-2}}{(j-2)!}\gamma_i) = 0; \ \ j = 2, 3, .., p+1. \quad (2.14)$$

And now consider the explanation of the condition $k \geq 2$.

From the first two equations of the system (2.12) or (2.14), we see that $\lambda = 1$ is the double root of the polynomial $\rho(\lambda)$. Noted that this condition is necessary for convergence of the method (2.13), therefore $\rho(\lambda)$ can be written as:

$$\rho(\lambda) = (\lambda - 1)^2(\overline{\alpha}_{k-2}\lambda^{k-2} + \overline{\alpha}_{k-3}\lambda^{k-3} + \overline{\alpha}_1\lambda + \overline{\alpha}_0). \qquad (2.15)$$

By taking into account this in the system of (2.14) receive that the amount of the unknowns equal to $4k+2$, but amount of the equations equal to $p+2$ and the received system will be linear nonhomogeneous, which will be have unique solution in the case $p = 2k$ and $\gamma_i = 0$ $(i = 0, 1, ..., k)$. From here it follows that $p \leq 2k$, if $\gamma_i = 0$ $(i = 0, 1, 2, .., k)$.

The system (2.14) is different from the systems (2.12), so as the system (2.14) is nonlinear and from the system (2.14) follows the system (2.12) in the case of $\gamma_i = 0$ $(i = 0, 1, ..., k)$. As was noted above in the system of (2.14) the amount of the unknowns equals to $4k + 4$, but the amount of the equations equals to $p + 2$. Here, also by using the properties of the first two equations, the amount of the equation can be taken as $p + 1$ and in this case, the homogeneous system (2.14) becomes a non-homogeneous system. In this case, receive that the system (2.14) can have the solution by which the methods will be constructed with the degree $p \leq 4k + 2$. And by construction the concrete stable methods with the degree $p = 3k + 3$ prove the existence of the stable methods of type (2.13) having the degree $p = 3k + 3$. Note that, if $\beta_i = 0$ $(i = 0, 1, ..., k)$, then there exist stable methods with the degree $p = 2k + 2$. For the shown this here have constructed stable methods with the degree $p = 2k + 2$ for the concrete values of k (see for example [9, 10, 11, 12], [22, 23, 24, 25, 26, 27, 28]). Let us note that the methods (18) are hybrid and advantages of which is known (see for example [9, 10, 11, 12], [20],[23],[27], [29, 30, 31]). It is not difficult to prove that if method (1.4) has the degree p, then there exists methods of type (1.4) with the degree $p \leq 3k + 1$. But if the method (1.4) has the degree p and stable then its degree satisfies the condition: $p \leq 2k+2$ (see for example [7],[8], [14], [29, 30, 31, 32]). By taking into account the above mentioned, one can take methods (1.4) and (2.13) (in the case, when $\beta_i = 0,\ i = 0, 1, ..., k$) as the equivalent in some means. It follows from here that the methods of type (2.13) can be taken as the better, so as they are more exact than the methods of type (1.4).

By comparison of all the above described advantages and disadvantages properties of the suggested methods, receive that the methods of type (2.13) have some advantages. Therefore, they can be taken as the perspective.

3. On Some Necessary Conditions for the Convergence of the Method (2.13)

For this aim, let us assume that method (2.13) is convergence and that by using the shift operator E can be written as the following:

$$\rho(E)y_n - h^2\delta(E)y''_n - h^2\gamma(E)y''_n = 0, \qquad (3.1)$$

here the polynomials are defined as:

$$\rho(\lambda) = \sum_{i=0}^{k} \alpha_i \lambda^i; \;\; \delta(\lambda) = \sum_{i=0}^{k} \beta_i \lambda^i; \;\; \gamma(\lambda) = \sum_{i=0}^{k} \gamma_i \lambda^{i+\nu_i}; \;\; E^{i+\nu_i}(y_n) = y_{n+i+\nu_i}.$$

From the equality of (3.1) the latter can be written:

$$\rho(E)y(x) = O(h^2). \tag{3.2}$$

From here, it follows that $\lim_{h\to 0} \rho(E) = \rho(1)$. By taking into account this, receive that $\rho(1)y(x) = 0$ and if use $y(x) \neq 0$, then it follows that $\lambda = 1$ is the root of the polynomial $\rho(\lambda)$. By taking into account $\rho(1) = 0$, the equality (3.1) can be written as:

$$\rho_1(E)(E-1)y_n - h^2(\delta(E) + \gamma(E))y''_n = 0, \tag{3.3}$$

Here, $\rho_1(\lambda) = \frac{\rho(\lambda)}{\lambda - 1} = \frac{\rho(\lambda)-\rho(1)}{\lambda-1}$. From here for $\lambda \to 1$ receive that $\rho_1(1) = \rho'(1)$.
By using this, the equality (3.3) can be written in the following form:

$$\frac{\rho'(E)(y_{n+1} - y_n)}{h} = h(\delta(E) + \gamma(E))y''_n. \tag{3.4}$$

Here, by passing to the limit for $h \to 0$, receive:

$$\rho'(1)y'(x) = h(\delta(E) + \gamma(E))y''(x), \tag{3.5}$$

here $x = x_0 + nh$ is fixed, point.

By assumption, the method (2.13) is convergence. Therefore, from the equality of (3.5) receive:

$$\rho'(1) = 0. \tag{3.6}$$

By using asymptotic equality (3.2) and the equality of (3.6), receive that

$$\rho(1) = \rho'(1) = 0.$$

From here it follows that $\lambda = 1$ is the double root for the $\rho(\lambda)$.

By using the condition (3.6) in the equality of (3.5) receive:

$$\frac{\rho(E)(y'_{n+1} - y'n)}{h} = (\delta(E) + \gamma(E))y'_n, \;\; \rho''(\lambda) = \lim_{\lambda\to 1} \frac{\rho'(\lambda) - \rho'(1)}{\lambda - 1}$$

(from $h \to 0$ it follows $E \to I$, here $E -$ is the single operation).

By passing to limit for $h \to 0$, receive the following:

$$\rho''(1) = \delta(1) + \gamma(1).$$

It follows from here that, if method (2.13) is convergence then the following take place:

$$\rho(1) = 0, \;\; \rho'(1) = 0, \quad \rho''(1) = \delta(1) + \gamma(1) \neq 0. \tag{3.7}$$

Thus receive that if method (2.13) is convergence, then $\lambda = 1$ must be double root for the polynomial $\rho(\lambda)$ and the condition $\rho''(1) = \delta(1) + \gamma(1) \neq 0$ or $p \geq 1$ must be done.

Remark 3.1. Let us consider the construction and application of some concrete stable methods of type (2.13). For this aim to put $k = 2$. In this case for $\beta_i = 0$ $(i = 0, 1, ..., k)$ from the (2.13) can be received the following method, which is stable and has the degree $p = 6$ $(p_{max} = 2k + 2)$:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(5y''_{n+1-\gamma} + 14y''_{n+1} + 5y''_{n+1+\gamma})/24, \ \gamma = \frac{\sqrt{10}}{5}. \qquad (3.8)$$

For application of the method to solve some problems we must propose any methods to calculate of the values $y_{n+1\pm\gamma}$ with the transaction error $O(h^6)$. For this, let us consider the following method:

$$y_{n+1+\alpha} = 2y_n - y_{n+1-\alpha} + \alpha^2 h^2(\hat{y}''_{n+1+\alpha} + 10\hat{y}''_{n+1} + \hat{y}''_{n+1-\alpha})/12, \qquad (3.9)$$

one can calculated the values $y_{n+1+\alpha}$ with the error $lte = O(h^6)$. For the simplification of the above mentioned sequence of the methods, the hybrid method let us construct in the form:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(4y''_{n+\gamma_2} + y''_{n+1} + 4y''_{n+\gamma_0})/9,$$
$$\gamma_0 = 1 - \sqrt{3}/4, \ \ \gamma_1 = 1 + \sqrt{3}/4, \qquad (3.10)$$

which is stable and has the degree $p = 4$. For the calculation of the values $y_{n+\gamma_2}$ and $y_{n+\gamma_0}$ with the order of exactness $O(h^4)$, here proposed to use the following formulas:

$$y_{n+\gamma_0} = y_{n+1} - \tfrac{\sqrt{3}}{4}(y_{n+1} - y_n) + \tfrac{36+7\sqrt{3}}{384}h^2 y''_{n+1} - \tfrac{61\sqrt{3}}{384}h^2 y''_n,$$
$$y_{n+\gamma_2} = y_{n+1} + \tfrac{\sqrt{3}}{4}(y_{n+1} - y_n) + \tfrac{36-7\sqrt{3}}{384}h^2 y''_{n+1} + \tfrac{61\sqrt{3}}{384}h^2 y''_n. \qquad (3.11)$$

For the comparison of Störmer-Verlet and (2.13) methods, let us consider the construction of the stable methods of type (2.13) in the case $k = 2$. In this case by using the solution of the above-mentioned systems of the algebraic equation, one can construct stable methods with the degree $p \leq 9$. But here we have constructed the stable method with the degree $p = 8$, which can be written as:

$$y_{n+2} = 2y_{n+1} - y_n + h^2(19y''_{n+2} + 870y''_{n+1} + 19y''_n)/1740+$$
$$+h^2(1323y''_{n+1-\gamma} + 58y''_{n+1} + 1323y''_{n+1+\gamma})/5655, \quad \gamma = \sqrt{13}/42. \qquad (3.12)$$

As is seen from the above-constructed methods, it is possible to construct more exact methods by using the methods of type (2.13).

Let us note that similar methods have been constructed by some authors for solving some different problems (see for example [22, 23, 24, 25]).

And now let us discuss the estimation of the error received in application of method (2.13). Let us suppose that the method of (2.13) has the degree of p and is stable. For the estimation of the error in the method (2.13), let us in the equality of (3.1) to change the approximate values of the solution of the problem (1.1) by its corresponding exact values. In this case the equality of (3.1) can be written as following:

$$\rho(E)y(x_n) - h^2\delta(E)y''(x_n) - h^2\gamma(E)y''(x_n) = R_n, \qquad (3.13)$$

here, $R_n = O(h^{p+2})$, $h \to 0$.

By subtracting equality (3.1) from the equality (3.13) receive:

$$\rho(E)\varepsilon_n - h^2\delta(E)\varepsilon''_n - h^2\gamma(E)\varepsilon''_n = R_n. \tag{3.14}$$

Here, $\varepsilon_n = y(x_n) - y_n$, $\varepsilon''_n = y''(x_n) - y''_n$. By using the condition $\rho(1) = 0$ (see equality of (3.7)) the equality (3.14) can be written as following:

$$\frac{\rho(E) - \rho(1)}{E - 1}(\varepsilon_{n+1} - \varepsilon_n)h^{-1} - h(\delta(E) - \gamma(E))\varepsilon''_n = \frac{R_n}{h}. \tag{3.15}$$

By taking into account that

$$\lim_{\lambda\to 1}(\rho(\lambda) - \rho(1))/(\lambda - 1) \lim_{h\to 0}(\varepsilon_{n+1} - \varepsilon_n)/h = \rho'(1)\varepsilon'_n,$$

from the equality of (35) for the case $h \to 0$, receive:

$$\rho'(1)\varepsilon'_n - h(\delta(E) - \gamma(E))\varepsilon''_n = R_n h^{-1}.$$

Let us denote ε'_n by the $\overline{\varepsilon_n}$, then without breaking the generality, the latter can be written:

$$\varepsilon_{n+1} = \varepsilon_n + h\overline{\varepsilon}_n. \tag{3.16}$$

From here, receive the following estimation:

$$\varepsilon_{n+1} = \varepsilon_n + h\overline{\varepsilon}_n = \varepsilon_{n-1} + h\overline{\varepsilon}_{n-1} + h\overline{\varepsilon}_n = \varepsilon_{n-2} + h\overline{\varepsilon}_{n-2} + h\overline{\varepsilon}_{n-1} + h\overline{\varepsilon}_n.$$

By continuing this process receive the following:

$$\varepsilon_{n+1} = \varepsilon_n + h\overline{\varepsilon}_n$$

or

$$\varepsilon_{n+1} = \varepsilon_0 + h\sum_{j=0}^{n}\overline{\varepsilon}_n.$$

From the last equality it follows that:

$$|\varepsilon_{n+1}| \le h(n+1)\overline{\varepsilon}; \;\; \overline{\varepsilon} = \max_n |\varepsilon_n|. \tag{3.17}$$

Similarly estimation for the method of (3.16) has been received by some authors in the investigation of the multistep methods. And have shown that if the method receiving for the magnitude $\overline{\varepsilon}_n$ is stable and has the degree of p ($\lambda_1 = 1$ is not multiple), then following takes place (see for example [13], [26]):

$$\max_n |\varepsilon_n| = O(h^p), \;\; h \to 0.$$

From this rate of approaches it follows that method (2.13) has the degree of p. This result one can be received in another way. It is known that the solution of the nonhomogeneous finite-difference equation by using the (3.14) can be presented as following $\varepsilon_m = \varepsilon_m^{(1)} + \varepsilon_m^{(2)}$, here $\varepsilon_m^{(1)}$ is the general solution of the corresponding homogeneous equation, but $\varepsilon_m^{(2)}$ is one of partial solution of the equation of (3.14). It is not difficult to understand that $\lim_{h\to 0}\varepsilon_m^{(2)} = 0$ is satisfied. Let us noted that the general solution of corresponding equation to (3.14) can be presented as following (if there is $\lambda_i \ne \lambda_j$ for $i \ne j$):

$$\varepsilon_m^{(1)} = C_1\lambda_1^m + C_2\lambda_2^m + C_3\lambda_3^m + ... + C_k\lambda_k^m \tag{3.18}$$

By assumption receive that the roots of the polynomial $\rho(\lambda)$ satisfies the next condition, $\lambda_1 = 1$ is double root, λ_i $(i = 3, 4, .., k)$ satisfies the condition of stability (the roots λ_i $(i = 3, 4, .., k)$) that is lies in the unite circle on the boundary of which there is not multiple roots. In the case $m = lh^{-1}$, from the equality of (3.18) receive that the magnitude $\varepsilon_m^{(1)}$ can be presented as follows:

$$h\varepsilon_m^{(1)} = c_1 h + c_2 l + h(c_3\lambda_3^m + .. + c_k\lambda_k^m).$$

It follows that $h\varepsilon_m^{(1)}$ is bounded for $h \to 0$.

From here it is reported that in order for the rate of approaches for the multistep methods to be equal to p, the polynomial $\rho(\lambda)$ must have the double root $\lambda = 1$. For the comparison of multistep methods, let us to consider following multistep multiderivative methods (MMM):

$$\sum_{i=0}^{k} \alpha_i y_{n+i} = \sum_{j=1}^{s} h^j \sum_{i=0}^{k} \beta_i^{(j)} y_{n+i}^{(j)}. \tag{3.19}$$

Note that the multiplicity of the root $\lambda_1 = 1$ for the method (2.13) equals 2 (two), but for the method (2.1) the root $\lambda_1 = 1$ is single. Let us note that this condition is the necessary and sufficient for the convergence of the method (2.13) (if $j = 2$, but for the method (2.13), if $l = 1$) to the solution of the following initial-value problem:

$$y^{(j)}(x) = f(x, y),\ y(x_0) = y_0,\ y^{(\nu)}(x_0) = y_0^{(\nu)},\ (\nu = 0, 1, 2, .., j-1).$$

Remark 3.2. Let us consider the following equalities:

$$\rho'(E)y'_n - h(\delta(E) + \gamma(E))y'_n = 0,$$

or

$$\rho(E)y_n - h^2(\delta(E) + \gamma(E))y''_n = 0.$$

If $\rho'(1) = 0$, then from here receive:

$$((\rho'(E) - \rho'(1))/(E-1))(y'_{n+1} - y'_n)/h - (\delta(E) + \gamma(E))y''_n = 0.$$

This equality can be written as:

$$\rho'(E)y''_n - (\delta(E) + \gamma(E))y''_n = 0.$$

It follows from here that $\rho''(1) = \delta(1) + \gamma(1)$.

These equalities for the following method, which corresponds to a method (2.1):

$$\sum_{i=0}^{k} (\alpha_i y_{n+i} - h\beta_i y'_{n+i}) = 0$$

can be written as:

$$\rho(E)y_n - h\delta(E)y'_n = 0.$$

It is not difficult to prove that $\rho(1) = 0$. In this case, from here the latter can be written:

$$\rho'(E)y'_n - \delta(E)y'_n = 0,$$

or

$$\rho'(1) = \delta(1).$$

If $\lambda = 1$ is the double root for the polynomial $\rho(\lambda)$, then $\delta(1) = 0$ and receive that the polynomials $\rho(\lambda)$ and $\delta(\lambda)$ have the common factor different from constant, which can be written as: $\phi(\lambda) = \lambda - 1$. This is corresponding to equality $\rho'(1) = \delta(1)$, where ($\lambda = 1$ is the double root). In this case the general solution of the corresponding homogeneous finite difference equation which corresponds to the method of (2.1), can be written as:

$$y_m = c_1 + c_2 m + c_3 \lambda_3^m + .. + c_k \lambda_k^m.$$

As follows from here, the proposed method is not convergence, because the polynomials $\rho(\lambda)$ and $\delta(\lambda)$ have the common factor different from the constants. This result has been received by Dahlquist, which follows from our result as the partial case. Therefore, the result received here can be taken as the development of Dahlquist's result. It follows from here that the solution of homogeneous finite – difference equations with constant coefficients will be unbounded if $\lambda = 1$ is double root for the polynomial $\rho(\lambda)$. In this case, the method (2.1) is not convergent. By the above described way, prove that if the method (2.13) is convergence, then its coefficients satisfies the following conditions:

A. Coefficients α_i, β_i, γ_i, ν_i $(i = 0, 1, .., k)$ some real numbers, moreover $\alpha_k \neq 0$.

B. Characteristic polynomials $\rho(\lambda)$, $\delta(\lambda)$ and $\gamma(\lambda)$ have not common factor different from constant.

C. The condition $\delta(1) + \gamma(1) \neq 0$ and $p \geq 1$ are satisfied.

It is not difficult to prove that all the methods, which have been constructed here, obeys the above described law. Let us note that the conditions A, B and C for the method of (2.1) can be written as follows (see [19]):

A. The coefficients α'_i, β'_i $(i = 0, 1, .., k)$ are real numbers and $\alpha'_k \neq 0$.

B. Characteristic polynomials $\overline{\rho}'(\lambda)$ and $\overline{\delta}'(\lambda)$ have not common factor different from constant.

C. $\overline{\delta}(1) \neq 0$ and $p \geq 1$.

Here $\overline{\rho}'(\lambda) \equiv \sum_{i=0}^{k} \overline{\alpha}'_i \lambda^i$, $\overline{\delta}'(\lambda) \equiv \sum_{i=0}^{k} \overline{\beta}'_i \lambda^i$.

4. Numerical results.

And now let us consider the application of some specific methods to solving model problems.

For the illustration of the results receiving here, let us applied Störmer-Verlet method and the methods (3.8), (3.10) to solve following problem:

$$y'' = (2 + 4x\lambda)exp(\lambda x) + \lambda^2 y(x), \; y(0) = 0, \; y'(0) = 0, \; 0 \leq x \leq 1, \qquad (4.1)$$

The exact solution of this problem can be presented as following:

$$y(x) = x^2 exp(\lambda x).$$

Note that the problem (4.1) was recently investigated by some authors using any modifications of midpoint and Euler's methods and some symmetric methods. In the work [34], above-mentioned has been applied to solve systems of ODE of the first and second order, where they have received interesting results.

A well known method is chosen here: the Störmer method (with step size h) and method Störmer 1 (with the step size $h/2$). These methods have different properties. Let us note that the solution of the problem (4.1) also has different properties. For example, if the argument x increases according to the solution, it also increases for the $\lambda > 0$, but for the $\lambda < 0$ the solution will decrease. Taking into account this, here have considered the cases when λ and h get different values. Note that here methods Störmer and Störmer 1 applied to solve problem (4.1) and the following:

$$y'' = \lambda^2 y(x),\ y(0) = 1,\ y'(0) = \lambda. \tag{4.2}$$

In this case receiving results denoted through Störmer and Störmer 1. The exact solution of the problem (4.2) can be presented as the $y(x) = exp(\lambda x)$. Note that all the methods in application of solving problem (4.2) give almost the same results. Therefore the results received in the application of the above named methods to solve problem (4.1) are not presented here. Note that in the construction of the algorithms, which applied to solve problem (4.1) have used some recommendations from the work [33]. In tables 1-4 have been tabulated the results received in solving problems (4.1) for the cases: $h = 0.1$; $h = 0.05$ and $\lambda = \pm 1$; $\lambda = \pm 5$.

Table 1. Results for $h = 0.1$ and $\lambda = 1$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	5.1E-08	1.4E-07	8.0E-12	3.1E-09
0.4	3.4E-07	9.5E-07	5.3E-11	2.0E-08
0.6	9.3E-07	2.6E-06	1.4E-10	5.4E-08
0.8	1.9E-06	5.5E-06	3.0E-10	1.1E-07
1.0	3.5E-06	9.8E-06	5.3E-10	2.0E-07

Table 2. Results for $h = 0.1$ and $\lambda = -1$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	3.8E-08	1.1E-07	6.2E-12	2.5E-09
0.4	2.1E-07	6.0E-07	3.4E-11	1.4E-08
0.6	4.8E-07	1.4E-06	8.0E-11	3.4E-08
0.8	8.3E-07	2.4E-06	1.4E-10	6.1E-08
1.0	1.2E-06	3.5E-06	2.0E-10	9.6E-08

From here the results received by the method (3.8) are better. Note that method (3.8) is more accurate than others. As is known if $\lambda < 0$, then the solution of the investigated problem will satisfy the condition $\lim_{x\to\infty} y(x) = 0$. Therefore, the following tables are contained by the results obtained for $h = 0.1$ and $\lambda = \pm 5$.

Table 3. Results for $h = 0.1$ and $\lambda = 5$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	5.6E-05	1.5E-04	2.1E-07	7.2E-05
0.4	6.0E-04	1.7E-03	2.1E-06	7.6E-4
0.6	2.9E-03	8.2E-03	10.0E-06	3.7E-3
0.8	1.1E-02	3.2E-02	3.7E-05	1.5E-2
1.0	1.2E-02	1.2E-01	1.3E-04	5.2E-2

Table 4. Results for $h = 0.1$ and $\lambda = -5$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	1.4E-05	4.0E-05	5.8E-08	2.6E-05
0.4	5.7E-05	1.6E-04	2.5E-07	1.5E-04
0.6	1.1E-04	3.0E-04	4.7E-07	4.7E-04
0.8	1.6E-04	4.5E-04	7.1E-07	1.3E-03
1.0	2.1E-04	6.0E-04	9.4E-07	3.6E-03

According to the results of this table, results received by the method of (3.8) are better. To confirm this let us consider the decreasing step-size.

Table 5. Results for $h = 0.05$ and $\lambda = 1$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	4.7E-09	1.3E-08	1.8E-13	2.8E-10
0.4	6.8E-08	6.8E-08	9.4E-13	1.4E-09
0.6	6.3E-08	1.8E-07	2.4E-12	3.7E-09
0.8	1.3E-07	3.6E-07	4.9E-12	7.4E-09
1.0	2.2E-07	6.3E-07	8.5E-12	1.3E-08

Table 6. Results for $h = 0.05$ and $\lambda = -1$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	3.7E-09	1.0E-08	1.5E-13	2.4E-10
0.4	1.6E-08	4.5E-08	6.4E-13	1.7E-09
0.6	3.4E-08	9.7E-08	1.4E-12	2.4E-09
0.8	5.7E-08	1.6E-07	2.3E-12	4.2E-09
1.0	8.4E-08	2.4E-07	3.5E-12	6.5E-09

If we compare the results received by the methods of (3.10) and Störmer then we obtain that the method (3.10) gives the best result, but the results received by the Störmer 1 are better than the results received by method (3.10).

Table 7. Results for $h = 0.05$ and $\lambda = 5$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	4.8E-06	1.4E-05	4.5E-09	6.6E-06
0.4	4.3E-05	1.1E-04	3.6E-08	5.5E-05
0.6	1.8E-04	5.3E-04	1.6E-07	2.6E-04
0.8	7.3E-04	2.1E-03	6.0E-07	9.9E-04
1.0	2.6E-03	7.3E-03	2.0E-06	3.5E-03

Table 8. Results for $h = 0.05$ and $\lambda = -5$.

x	Method (3.10)	Method Störmer	Method (3.8)	Method Störmer 1
0.2	1.5E-06	4.2E-06	1.5E-09	2.9E-06
0.4	4.9E-05	1.4E-05	5.2E-09	1.4E-05
0.6	8.7E-05	2.4E-05	9.4E-09	4.1E-05
0.8	1.2E-05	3.5E-05	1.4E-08	1.1E-04
1.0	1.6E-05	4.6E-05	1.8E-08	3.1E-04

From the above-mentioned it follows that the results received by the method (3.8) are better in all the cases. Note that, here have used some results from the works [35] and [36] in the construction of predictor-corrector methods and used some connections between the Cauchy problem and Volterra integral equations.

The Cauchy problem for matrix factorization of the Helmholtz equation are considered in papers [36]-[38]. As is known there are methods applied to solve this equation by using nonlinear methods. For example in [39] proposed to use a finite difference Scheme based on the Rosenbrock method. Note that the Rosenbrock method is nonlinear but the finite-difference method is linear. By taking into account this, here by using linear multistep method of hybrid type, have constructed a method which can be written as the method of (2.13). Note that the method (2.13) in [40] has been applied to solve the initial value problem for ODE and receive a good result. And in the work [41] had considered comparing the Rosenbrock method with the advanced method. In considering the case the advanced method gives the best result, than the Rosenbrock.

Note that there are different ways to construct multistep methods of the hybrid type for solving the different problems (see for example [42, 43, 44, 45, 46, 47]).

5. Conclusion

Hybrid methods, which have been constructed here, are simple and have a higher order of accuracy. Here, have investigated some of generalization of the Störmer – Verlet method and considered the application of that to solve initial-value problem for ODE of the second order with the special structure, from which one can be receive many known equations as Scrhrödinger, Shturm-Liouville and others. As is known there are some classes of methods constructed for solving the initial-value problem for ODEs of the second order. One of the effective methods for solving problem (1.1) is method (1.4), so that is more accurate than the others. Here, it has been proven that one of the efficient methods for solving problem (1.2) and (2.8) is the hybrid method of Störmer-Verlet type, which can be received from the method (2.13) as the partial case.

By using the Dahlquist laws, we can see the method (1.4) is more exact than the others. But here, have shown that the hybrid method is more exact than the method of (1.4). By taking into account this, here for solving the considering problems with the high order of accuracy, suggested to investigate hybrid methods, which constructed by using formula (2.13) and have proved that if method defined by the formula (2.13) is stable and has the degree of p, then their degree will satisfy the condition as $p \leq 3k + 3$. It follows from here that the hybrid methods are more exact than the others, therefore these methods are perspective. Here have defined the necessary conditions imposed on the coefficients of the method (2.13), which is very important for the application of these methods to solve some problems and also for defining the necessary conditions for the convergence of the method (2.13).

Note that depending on the properties of the solution of the investigated problem, the methods with fractional step-size can give the results, which are worse than the results obtained by the nonfractional methods. It is not difficult to obtain that methods with the fractional step-size are also a part of the class of hybrid methods. As follows from here, obtaining results here has a significant advantage in solving many applied physical problems including above-mentioned problems such as Scrhrödinger, Shturm-Liouville etc. As is known often there is a need to study the numerical solution of integral equations of the Volterra type by taking into account connection between ordinary differential equations of the Volterra type. The papers [48, 49, 50, 51, 52, 53, 54] explored the application of hybrid methods to solve the Volterra integral equations, where the stability region for some methods was studied.

Acknowledgment. The authors wish to express their thanks to academicians Telman Aliyev and Ali Abbasov for their suggestion to investigate the computational aspects of our problem and for their frequent valuable suggestions. This work was supported by the Science Development Foundation under the President of Republic of Azerbaijan - Grant AEF-MCG-2022-1(42)-12/4/1-M-4 (for Vagif Ibrahimov). 2020-2025, Hubei ChuTian Scholar Funding, China (For Xiao-Guang Yue).

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