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LINEAR SECOND ORDER ORDINARY DIFFERENTIAL EQUATIONS WITH ONE SINGULAR POINT FROM THE VIEW POINT OF DIVISION BY ZERO CALCULUS

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Abstract. By the Uchida's basic idea, the products of the first order and second order normal solutions (Uchida's hyper exponential functions) generate the fully general solutions of the second order linear equations. However, the second order normal solutions are known for very few concrete functions only. Meanwhile, of course, the first order normal solutions are general and trivial. For the importance of the second order linear ordinary differential equations with a singularity we would like to consider some general and concrete ordinary differential equations. On this line, by the concept of division by zero calculus, we would like to consider the typical special singularities. For the first order differential equation, we consider the typical cases of the function $y = |x-a|^\nu$ and for the second order differential equations $y = (x-a)^\nu$ that satisfy $y' = \frac{\nu}{(x-a)} y$ and $y'' = \frac{\nu(\nu-1)}{(x-a)^2} y$, respectively.

David Hilbert:

The art of doing mathematics consists in finding that special case which contains all the germs of generality.

Oliver Heaviside:

Mathematics is an experimental science, and definitions do not come first, but later on.

1. Introduction

In order to represent the paper in a self contained manner, we set first simply the needed basic materials.

K. Uchida ([23]) has a long love for the solutions of the differential equations

$$\frac{d^n y}{dx} = f(x)y$$

and he is appointing out the importance of the solutions. He called the solutions *hyper exponential functions* (*Uchida's hyper exponential functions*). He considered the solutions for some functions $f(x)$ and derived many beautiful computer graphics with their elementary properties ([24]). We see the few concrete solutions from

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[16] and [24] for $n = 2$. Of course, the case $n = 1$ is trivial and the $n > 3$ cases are rare examples, and the case $n = 2$ is important.

His fundamental results are stated as follows:

Proposition 1.1. *For the solutions*

$$u' = f(x)u$$

and

$$v'' = g(x)v,$$

the product $y = uv$ is the solution of the differential equation of

$$y'' + A(x)y' + B(x)y = 0 \tag{1.1}$$

with

$$A(x) = -2f(x)$$

and

$$B = f(x)^2 - f'(x) - g(x).$$

The general solution of (1.1) is given by u and linearly independent solutions v_1 and v_2 in the form

$$y = u(C_1 v_1 + C_2 v_2).$$

For the general inhomogenous equation

$$y'' + A(x)y' + B(x)y = Q(x), \tag{1.2}$$

we have a special solution

$$y = u(x)v_2(x)\int u(t)^{-1}v_1(t)Q(t)dt - u(x)v_1(x)\int u(t)^{-1}v_2(t)Q(t)dt,$$

if the integrals exist.

Note that in Proposition 1.1, the functions f and g can be solved conversely in terms of the functions A and B;

$$f(x) = -\frac{A(x)}{2}, \quad g(x) = \frac{A(x)^2}{4} + \frac{A'(x)}{2} - B(x). \tag{1.3}$$

Next, we state the essence of the division by zero calculus.

For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n, \tag{1.4}$$

we will define

$$f(a) = C_0. \tag{1.5}$$

For the correspondence (1.5) for the function $f(z)$, we will call it **the division by zero calculus**. By considering derivatives in (1.4), we **can define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n!C_n.$$

Here, however, we will use the general definition of the division by zero calculus:

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{1.6}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as $1/0 = 0$ and $0/0 = 0$, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense in [20].

For the case of non differentiable functions, we shall define the values at a singular point by using the formal results $1/0 = 0$ and $0/0 = 0$. In this case, we shall consider the results as conventions and we will check the results. At this moment, for odd function satisfying $f(-x) = -f(x)$, the result $f(0) = 0$ may be natural from $f(0) = -f(0)$.

The first order normal solutions are trivial, but we will introduce typically interesting solutions with typical singularities.

First, for the function $y = |x|$ we obtain:

From the expression

$$y := \exp\left(\int^x \frac{dt}{t}\right) = \exp\left(\log|x|\right) = |x|,$$

$$y' = |x|\frac{1}{x} = \frac{1}{x}y$$

and

$$y'(0) = 0$$

by the division by zero calculus. Next, for some general case of the function

$$y = \frac{1}{m}\left(|x| + x\right),$$

we have

$$y' = \frac{1}{m}\left(\frac{|x|}{x} + 1\right)$$

and

$$y' = \frac{1}{x}y.$$

Then, we obtain

$$y'(0) = \frac{1}{m}.$$

For a C_1 function $y = f(x)$ except for an isolated point $x = a$ having $f'(a-0)$ and $f'(a+0)$, we shall introduce its natural differential coefficient at the singular point $x = a$. Surprisingly enough, the differential coefficient is given by the division by zero calculus and it will give the gradient of the natural tangential line of the function $y = f(x)$ at the point $x = a$.

We obtain the very pleasant theorem

Theorem 1.1. *At the point $x = a$, we introduce the definition*

$$f'(a) = \frac{1}{2}\left(f'(a-0) + f'(a+0)\right).$$

Then, $f'(a)$ has the sense of the gradient of the natural tangential line at the point $x = a$ of the function $y = f(x)$ and it is given by the division by zero calculus at the point in the sense that: For the function

$$F(x) = \frac{f'(a-0) + f'(a+0)}{2}\left(|x-a| + (x-a)\right) - f'(a-0)|x-a|,$$

$$F'(a) = f'(a)$$

in the sense of the division by zero calculus.

Anyhow, the definition

$$y' = |x|\frac{1}{x}$$

may be understood as the natural one as the derivative containing the origin $x = 0$ by the division by zero calculus.

In particular, note that the division by zero calculus is beyond continuity and differentiability of a function to determine the value at a singular point.

Indeed, consider the function

$$f(x) = \begin{cases} x^2 \sin \frac{1}{x}, \quad (x \neq 0) \\ 0 \quad (x = 0). \end{cases} \tag{1.7}$$

Then, we obtain also the result $f(0) = 0$ by the division by zero calculus. However, for its differential

$$f'(x) = \begin{cases} 2x \sin \frac{1}{x} - \cos \frac{1}{x}, \quad (x \neq 0) \\ 0 \quad (x = 0), \end{cases} \tag{1.8}$$

this function is differentiable at $x = 0$, however, we have $f'(0) = 1$ by the division by zero calculus.

In the division by zero calculus, we consider mathematics except singular points first, and next we consider the values at isolated singular points by the division by zero calculus.

For the first order normal solutions with singularities (for the negative ν), we shall consider the functions

$$y = U_\nu(x) = |x-a|^\nu, \quad \nu \neq 0 \tag{1.9}$$

satisfying

$$y' = \frac{\nu}{x-a} y. \tag{1.10}$$

Next we will consider the second order normal solutions with singularities (for the negative ν). For the functions

$$y = V_{\nu,1}(x) = (x-a)^{\nu}, \tag{1.11}$$

we obtain the normal equations

$$y'' = \frac{\nu(\nu-1)}{(x-a)^2} y, \quad \nu \neq 0, 1. \tag{1.12}$$

Then we see that the functions

$$Y = V_{\nu,2}(x) = \frac{\nu(\nu-1)}{\mu(\mu-1)} (x-a)^{\mu} \tag{1.13}$$

are linearly independent solutions for $\nu \neq \mu$ ($\neq 0, 1$). In the general theory of Uchida, we shall use these fundamental solutions for the second order of the normal equation.

In order to consider some general linear second order ordinary differential equations, for the first order normal solution we shall consider the function for a continuous function $y = e(x)$

$$y = E(x) = \exp\left(\int^x e(t)dt\right)$$

satisfying

$$y' = e(x)y. \tag{1.14}$$

Meanwhile, for the second order normal solution we shall consider the function for non-vanishing C_2 function

$$y = S(x)$$

satisfying

$$y'' = \frac{S''(x)}{S(x)} y. \tag{1.15}$$

For a linearly independent solution, we shall consider the function

$$S_2(x) = S(x) \int^x \frac{dt}{S(t)^2}.$$

2. General Formulas

By the Uchida's results and the above concrete solutions, we obtain the following general formulas by assuming the existence of the integrals. Anyhow, we can determine the structure of linear second order ordinary differential equations having one singular pole as the solutions in some general situation.

I: *For*

$$A(x) = -2e(x)$$

and

$$B(x) = e(x)^2 - e'(x) - \frac{S''(x)}{S(x)},$$

we have for

$$u = E(x),$$
$$v_1 = S(x)$$

and

$$v_2 = S(x) \int^x \frac{dt}{S(t)^2},$$

we have the complete solution.

II: *For*

$$A(x) = -2\frac{\nu}{x-a}$$

and

$$B(x) = \frac{\nu^2 + \nu}{(x-a)^2} - \frac{S''(x)}{S(x)},$$

we have for

$$u = U_\nu(x),$$
$$v_1 = S(x)$$

and

$$v_2 = S(x) \int^x \frac{dt}{S(t)^2},$$

we have the complete solution.

III: *For*

$$A(x) = -2e(x)$$

and

$$B(x) = e(x)^2 - e'(x) - \frac{\nu(\nu-1)}{(x-a)^2},$$

we have for

$$u = E(x),$$
$$v_1 = V_{\nu,1}(x)$$

and

$$v_2 = V_{\nu,2}(x)$$

we have the complete solution.

IV: *For*

$$A(x) = -2\frac{\nu}{x-a}$$

and

$$B(x) = \frac{\nu^2 + \nu}{(x-a)^2} - \frac{\nu'(\nu'-1)}{(x-a)^2},$$

we have for

$$u = U_\nu(x),$$
$$v_1 = V_{\nu',1}(x)$$

and

$$v_2 = V_{\nu',2}(x),$$

we have the complete solution.

3. Singular Point and Fractional Power

Note that the Uchida's decomposition theorem is valid even for the case that u and v (so, A and B) have singular points, because the equalities (identities) used are done by using derivatives and formal algebraic identities. In particular, for integer order power and derivatives, there is no problem. However, for a fractional order ν when we consider that: for

$$y = V_{\nu,1}(x) = (x-a)^\nu, \tag{3.1}$$

we obtain the normal equations

$$y'' = \frac{\nu(\nu-1)}{(x-a)^2}y, \quad \nu \neq 0, 1, \tag{3.2}$$

we have the serious problem for the definition $(x-a)^\nu$. Here, we are considering mathematics on the real number field, however, for the real power (not integer) of ν we should restrict as $x - a > 0$ otherwise the definition of $(x-a)^\nu$ is unclear by its definition. In this case as well-known, its derivative is well-known as

$$((x-a)^\nu)' = \nu(x-a)^{\nu-1}, \quad \nu > 0, \nu < 0$$

and other formulas.

In this sense, the above results should be considered in this principle.

Meanwhile, for the result

$$\left(\frac{1}{x^n}\right)_{x=0} = 0$$

for non-zero integers n, this result still is valid for any non zero real number ν.

This property may be derived as follows:

For the Laplace transform of the function

$$\frac{t^{n-1}e^{-at}}{(n-1)!}, \quad n = 1, 2, 3, ...,$$

we have

$$\frac{1}{(s+a)^n}.$$

Then, for $s = -a$, by the division by zero calculus (DBZC), we have

$$\left[\frac{1}{(s+a)^n}\right]_{s=-a} = 0.$$

Then, how will be the corresponding Laplace transform

$$\int_0^\infty \frac{t^{n-1}e^{-at}}{(n-1)!}e^{at}dt = \int_0^\infty \frac{t^{n-1}}{(n-1)!}dt$$

? Note that this integral is zero, because infinity may be represented by 0. For many geometrical examples and analytical meanings, see the papers cited in the references. For example [1, 4, 15, 18].

Conversely, from this argument for the general function for any positive ν

$$\frac{\Gamma(\nu)}{(s+a)^\nu}$$

that is the Laplace transform of the function

$$t^{\nu-1}e^{-at},$$

we can derive the result

$$\left[\frac{\Gamma(\nu)}{(s+a)^\nu}\right]_{s=-a} = 0.$$

Indeed, since this result is not defined by DBZC for general positive ν, this result now was derived here, by this logic.

4. Many Second Order Normal Solutions

At this moment, it seems that the second order normal solutions are known only for few concrete examples as we see from [16] and at the same time, we know many linear independent solutions of the second order ordinary differential equations, as we see from the same hand book. From the general Proposition 1.1, we see that conversely from linear independent solutions of the second order ordinary differential equations, we can find linear independent normal second order solutions. Therefore, now we can find many and many kind normal solutions for the second order.

For example, for the differential equation

$$y'' + \left(2a + \frac{b}{x}\right) y' + a\left(a + \frac{b}{x}\right) y = 0, \tag{4.1}$$

we obtain the general solutions

$$y = \begin{cases} e^{ax}(C_1 + C_2 x^{1-b}), & b \neq 1 \\ e^{ax}(C_1 + C_2 \log|x|), & b = 1, \end{cases} \tag{4.2}$$

([16], page 222, 72). From

$$f(x) = -a - \frac{b}{2x},$$

we have the first order normal solution

$$y = e^{-ax}x^{-\frac{b}{2}}.$$

Therefore, for the functions

$$g(x) = \frac{b}{2x^2}\left(\frac{b}{2}\right),$$

we obtain the linear independent second order normal solutions

$$x^{\frac{1}{2}b}$$

and
$$x^{1-\frac{1}{2}b}.$$

Precisely, here, for non integer b, we are considering the equations for $x > 0$ and for the singular point at $x = 0$ we have to give the interpretation by the concept of the division by zero calculus as in the above.

Next, for the differential equation
$$y'' + \frac{n}{x}y' + \frac{b}{x^{2n}}y = 0, \quad n \neq 1,$$
we have the general solutions,
$$y = \begin{cases} C_1 \sin\left(\frac{\sqrt{b}}{n-1}x^{1-n}\right) + C_2 \cos\left(\frac{\sqrt{b}}{n-1}x^{1-n}\right), & b > 0 \\ C_1 \exp\left(\frac{\sqrt{-b}}{n-1}x^{1-n}\right) + C_2 \exp\left(\frac{\sqrt{-b}}{n-1}x^{1-n}\right), & b < 0 \end{cases} \tag{4.3}$$
([16], page 220, 65).

Then, from
$$f(x) = -\frac{n}{2x},$$
we have the first order normal solution
$$y = x^{-\frac{n}{2}}.$$
Therefore, for the function
$$g(x) = \frac{n}{4x^2}(n-2) - \frac{b}{x^{2n}},$$
we obtain the linear independent second order normal solutions
$$y = \begin{cases} x^{\frac{n}{2}} \sin\left(\frac{\sqrt{b}}{n-1}x^{1-n}\right), x^{\frac{n}{2}} \cos\left(\frac{\sqrt{b}}{n-1}x^{1-n}\right), & b > 0 \\ x^{\frac{n}{2}} \exp\left(\frac{\sqrt{-b}}{n-1}x^{1-n}\right), x^{\frac{n}{2}} \exp\left(\frac{\sqrt{-b}}{n-1}x^{1-n}\right), & b < 0. \end{cases} \tag{4.4}$$

For non integer power, we consider the results for positive $x > 0$ and at the singular point, we give the interpretation by the division by zero calculus.

Next, for the Euler equation
$$y'' + \frac{a}{x}y' + \frac{b}{x^2}y = 0,$$
we have the general solutions, for
$$\mu = \frac{1}{2}|(1-a)^2 - 4b|^{1/2},$$

$$y = \begin{cases} |x|^{\frac{1-a}{2}}(C_1|x|^{\mu} + C_2|x|^{-\mu}), & (1-a)^2 > 4b \\ |x|^{\frac{1-a}{2}}(C_1 + C_2 \log|x|), \quad b = 1, & (1-a)^2 = 4b \\ |x|^{\frac{1-a}{2}}(C_1 \sin(\mu \log|x|^{\mu} + C_2 \cos\mu \log|x|), & (1-a)^2 < 4b, \end{cases} \tag{4.5}$$
([16], page 226, 123).

Then, from

$$f(x) = -\frac{a}{2x},$$

we have the first order normal solution

$$y = x^{-\frac{a}{2}}.$$

Therefore, for the function

$$g(x) = \frac{1}{4x^2}(a^2 - 2a - 4b),$$

we obtain the linear independent second order normal solutions

$$y = \begin{cases} |x|^{\mu+\frac{1}{2}}, |x|^{-\mu+\frac{1}{2}}, \quad (1-a)^2 > 4b \\ |x|^{\frac{1}{2}}, |x|^{\frac{1}{2}} \log|x|, \quad b = 1, \quad (1-a)^2 = 4b \\ |x|^{\frac{1}{2}} \sin(\mu \log|x|), |x|^{\frac{1}{2}} \cos(\mu \log|x|), \quad (1-a)^2 < 4b. \end{cases} \tag{4.6}$$

5. Examples

We shall state examples in order to see the division by zero calculus at singular points.

- For the function

$$y = \frac{\log|x|}{|x|}. \tag{5.1}$$

 we have

$$y' = \frac{1}{x|x|}(1 - \log|x|) \tag{5.2}$$

 and

$$y(0) = y'(0) = 0.$$

- For the function

$$y = |x| \log|x|. \tag{5.3}$$

 we have

$$y' = \frac{|x|}{x}(1 + \log|x|) \tag{5.4}$$

 and

$$y(0) = y'(0) = 0.$$

- For the function

$$y = |x| \cosh x, \tag{5.5}$$

 we have

$$y' = \frac{|x|}{x} \cosh x - |x| \sinh x \tag{5.6}$$

 and

$$y(0) = y'(0) = 0.$$

- For the function
$$y = \frac{\cosh x}{|x|}, \tag{5.7}$$
we have
$$y' = \frac{1}{|x|}\left(\sinh x - \frac{\cosh x}{x}\right) \tag{5.8}$$
and
$$y(0) = y'(0) = 0.$$
- For the function
$$y = \frac{\sinh x}{|x|}, \tag{5.9}$$
we have
$$y' = \frac{1}{|x|}\left(\cosh x - \frac{\sinh x}{x}\right) \tag{5.10}$$
and
$$y(0) = y'(0) = 0.$$
- For the function
$$y = |x| \sin x, \tag{5.11}$$
we have
$$y' = \frac{|x|}{x}(\sin x + |x| \cos x) \tag{5.12}$$
and
$$y(0) = y'(0) = 0.$$

For the function
$$y = |x| \cos x, \tag{5.13}$$
we have
$$y' = \frac{|x|}{x}(\cos x - |x| \sin x) \tag{5.14}$$
and
$$y(0) = y'(0) = 0.$$
- For the function
$$y = \frac{\sin x}{|x|}, \tag{5.15}$$
we have
$$y' = \frac{1}{|x|}\left(\cos x - \frac{\sin x}{x}\right) \tag{5.16}$$
and
$$y(0) = y'(0) = 0.$$
- For the function
$$y = \frac{\cos x}{|x|}, \tag{5.17}$$
we have
$$y' = -\frac{1}{|x|}\left(\sin x \frac{\cos x}{x}\right) \tag{5.18}$$
and
$$y(0) = y'(0) = 0.$$

- For the function
$$y = \frac{1}{x} + \frac{1}{|x|}, \tag{5.19}$$
we have
$$y' = -\frac{1}{x}\left(\frac{1}{x} + \frac{1}{|x|}\right) = -\frac{1}{x}y \tag{5.20}$$
and
$$y(0) = y'(0) = 0.$$

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