International Journal of Division by Zero Calculus
Vol. 2 No. 1-2 (January-December, 2022)

Received: 16th April 2022 **Revised: 18th June 2022** **Accepted: 25th June 2022**

AN ICONIC CONFIGURATION OF WASAN GEOMETRY WITH $1/0 = 0$

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Abstract. We consider limiting figures of the configuration consisting of three mutually externally touching circles with a common tangent by division by zero $1/0 = 0$.

1. Introduction

Japanese mathematics in the Edo era (1603-1867) is called Wasan. Wasan mathematicians were interested in figures involving touching circles in the plane. In this paper, we consider one of the most frequently considered configuration in Wasan geometry consisting of three mutually externally touching circles with their common tangent (see Figure 1). We denote the configuration by $\mathcal{W}$, and consider limiting cases of $\mathcal{W}$ using division by zero $1/0 = 0$ ([1], [2]).

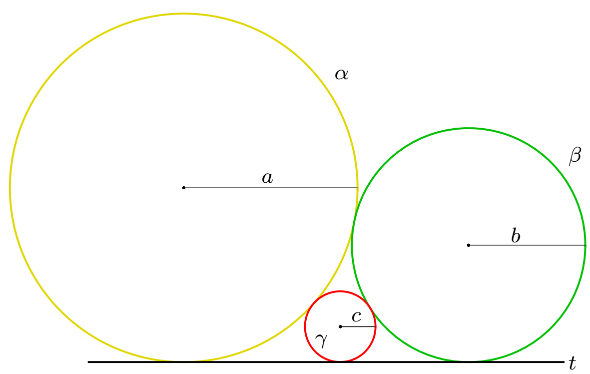

Figure 1: The configuration $\mathcal{W}$.

We fix one of the circles and the common tangent, while the other circles vary, and consider the case in which the radius of one of the varying circles equals 0. The configuration is uniquely determined, if the three circles have non-zero radii. However we will see that if one of the circles has radius 0, then the configuration is not uniquely determined.

We list the followings, which will be used in this paper.

Proposition 1. *If two externally touching circles of radii p and q touch one of their external common tangents at points P and Q, then $|PQ| = 2\sqrt{pq}$.*

2000 *Mathematics Subject Classification.* Primary 51M04; Secondary 01A27, 12E99, 41A58,
Key words and phrases. touching circles, limiting figure, division by zero.

Definition 1 ([1], [2])**.** $\frac{z}{0} = 0$ for any real number z.

The definition is referred to as *DBZ*. The next two theorems can be derived from DBZ.

Theorem 1 ([2], [3])**.** *Two orthogonal figures can be considered to touch.*

Theorem 2 ([2], [3])**.** *A line can be considered to be a circle of radius* 0*.*

2. The configuration

Let α, β and γ be mutually externally touching circles forming the configuration $\mathcal{W}$ of radii a, b and c ($c < a$ and $c < b$), respectively with an external common tangent t. We consider the following three cases:
(Case 1): We fix the circle α and the tangent t, while the circles β and γ vary, and consider the limiting configuration $\mathcal{W}$ in the case $b = 0$.
(Case 2): We fix the circle α and the tangent t, while the circles β and γ vary as in Case 1, and consider the limiting configuration $\mathcal{W}$ in the case $c = 0$.
(Case 3): We fix the circle γ and the tangent t, while the circles α and β vary, and consider the limiting configuration $\mathcal{W}$ in the case $b = 0$.

By Proposition 1, we have

$$\sqrt{ab} = \sqrt{bc} + \sqrt{ca}, \tag{2.1}$$

or

$$\frac{1}{\sqrt{c}} = \frac{1}{\sqrt{a}} + \frac{1}{\sqrt{b}}. \tag{2.2}$$

We use a rectangular coordinate system with origin at the point of tangency of the fixed circle and t so that the center of the circle β has coordinates $(2\sqrt{bf}, b)$, where f is the radius of the fixed circle. There are three choices which circles we fix, and for each of the three cases, there are two choices which radius of varying circles we consider to be 0. Hence there are six cases in all to consider the limiting cases. However we consider the three cases just listed above, since each of the remaining cases is symmetric to one of Cases 1, 2 and 3 about the y-axis.

3. Case 1

In this section, we fix the circle α and the tangent t, and consider the limiting configurations of $\mathcal{W}$ in the case $b = 0$ using a rectangular coordinates system with origin at the point of tangency of α and t so that the center of β has coordinates $(2\sqrt{ab}, b)$.

3.1. The circle β. The circle β has an equation

$$\beta(x, y) = \left(x - 2\sqrt{ab}\right)^2 + (y - b)^2 - b^2 = 0, \tag{3.1}$$

which can be arranged in three ways as follows:

$$(x^2 + y^2) - 4\sqrt{ab}x - 2b(y - 2a) = 0,$$
$$\frac{x^2 + y^2}{\sqrt{b}} - 4\sqrt{a}x - 2\sqrt{b}(y - 2a) = 0,$$

$$\frac{x^2+y^2}{b} - 4\sqrt{\frac{a}{b}}x - 2(y-2a) = 0.$$

Therefore $b = 0$ implies $x^2 + y^2 = 0$, $x = 0$ and $y - 2a = 0$ by DBZ. The last three equations represent the origin, the y-axis and the tangent of α parallel to t. We denote the three figures by the same symbols β_0, β_1 and β_2, respectively, and indicate in Figure 2 by green.

3.2. The circle γ. The circle γ has an equation

$$\gamma(x,y) = (x - 2\sqrt{ac})^2 + (y-c)^2 - c^2 = 0, \tag{3.2}$$

while by (2.1), we have

$$c = \frac{ab}{\left(\sqrt{a}+\sqrt{b}\right)^2}. \tag{3.3}$$

Substituting (3.3) in (3.2) and rearranging, we get:

$$\gamma(x,y) = \frac{a\gamma_0 + 2\sqrt{ab}\gamma_1 + b\gamma_2}{\left(\sqrt{a}+\sqrt{b}\right)^2},$$

where $\gamma_0 = x^2 + y^2$, $\gamma_1 = (x-a)^2 + y^2 - a^2$, and $\gamma_2 = (x-2a)^2 + (y-a)^2 - a^2$. Hence $\gamma(x,y) = 0$ implies

$$\begin{aligned} a\gamma_0 + 2\sqrt{b}\sqrt{a}\gamma_1 + b\gamma_2 &= 0, \\ \frac{a\gamma_0}{\sqrt{b}} + 2\sqrt{a}\gamma_1 + \sqrt{b}\gamma_2 &= 0, \\ \frac{a\gamma_0}{b} + \frac{2\sqrt{a}\gamma_1}{\sqrt{b}} + \gamma_2 &= 0. \end{aligned}$$

Therefore $b = 0$ implies $\gamma_0 = 0$, $\gamma_1 = 0$, and $\gamma_2 = 0$ by DBZ, which represent the origin, the circle of radius a and center of coordinates $(a, 0)$, and the circle of radius a and center of coordinates $(2a, a)$, respectively. We denote the three figures by the same symbols γ_0, γ_1 and γ_2, respectively, and indicate in Figure 2 by red. Notice that the circle γ_1 touches the circle α and the line t by Theorem 1.

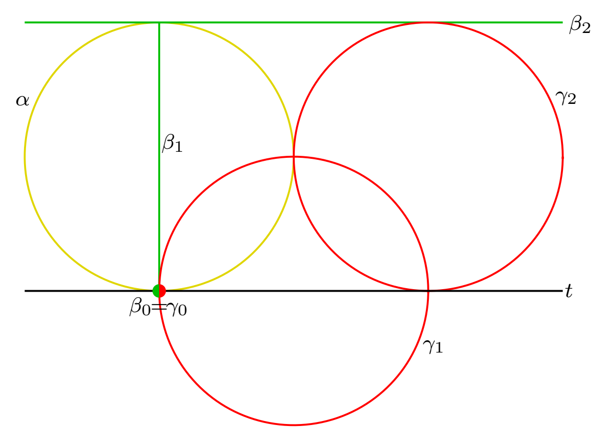

Figure 2: β and γ in the case $b = 0$.

3.3. The limiting configurations. We consider the possible combinations of the figures α, β_i and γ_j forming a limiting configuration of $\mathcal{W}$ in the case $b = 0$. Then the limiting configurations can be obtained if : $(\beta, \gamma) = (\beta_0, \gamma_0)$, (β_0, γ_1), (β_1, γ_0), (β_1, γ_1), (β_2, γ_2). If the three figures α, β_i and γ_j form a limiting configuration of $\mathcal{W}$, we explicitly denote it by $\mathcal{W}_{ij}$. Then we get five configurations $\mathcal{W}_{00}$, $\mathcal{W}_{01}$, $\mathcal{W}_{10}$, $\mathcal{W}_{11}$, $\mathcal{W}_{22}$. For the configurations $\mathcal{W}_{00}$ and $\mathcal{W}_{10}$ described in Figures 3 and 4, we get $b = c = 0$ by Theorem 2. Hence (2.1) holds, but (2.2) does not by DBZ. For the remaining three configurations $\mathcal{W}_{01}$, $\mathcal{W}_{11}$, $\mathcal{W}_{22}$ described in Figures 5, 6 and 7, we get $b = 0$ and $c = a$ by Theorem 2. Hence (2.1) does not hold, but (2.2) does by DBZ. Therefore *if we fix the circle α and the line t and consider the limiting configurations of $\mathcal{W}$ in the case $b = 0$, then we have the five limiting configurations just mentioned in the above, each of which satisfies exactly one of* (2.1) *and* (2.2).

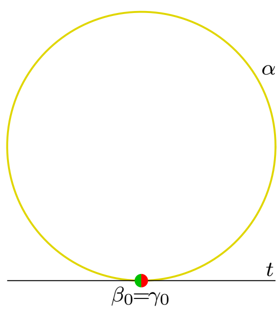

Figure 3: $\mathcal{W}_{00}$.

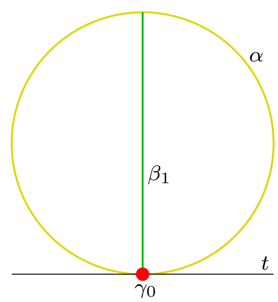

Figure 4: $\mathcal{W}_{10}$.

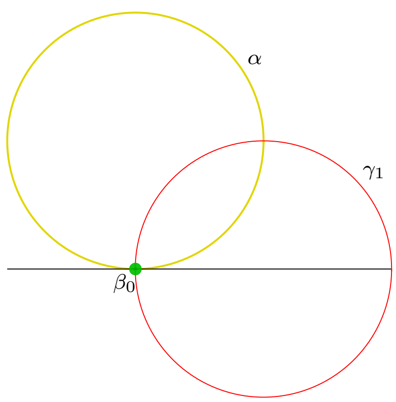

Figure 5: $\mathcal{W}_{01}$.

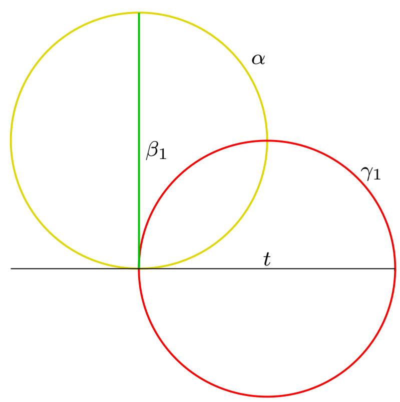

Figure 6: $\mathcal{W}_{11}$.

It is easy to find the configuration $\mathcal{W}_{00}$ indicated in Figure 3. The configuration $\mathcal{W}_{22}$ indicated in Figure 7 can be obtained as a limiting case when we consider the case in which the radius of β increases without limit. However notice that in the limiting case we have $b = 0$ by Theorem 2. Such a discontinuity can frequently be seen when DBZ occurs. Consider the function $f(x) = 1/x$ for example. Then we have $f(0) = 0$ by DBZ, while $\lim_{x\to\pm 0} f(x) = \pm\infty$. To find the other three configurations $\mathcal{W}_{10}$, $\mathcal{W}_{01}$ and $\mathcal{W}_{11}$ seems to be impossible without a consideration such like the one given in this section using DBZ.

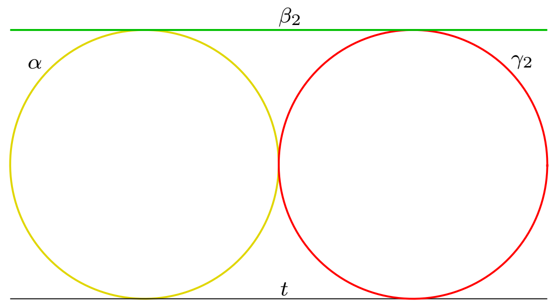

Figure 7: $\mathcal{W}_{22}$.

4. Case 2

We fix the circle α and the tangent t, and consider the limiting configurations of $\mathcal{W}$ in the case $c = 0$ using the same coordinates system as in Section 3. We consider the circle β similarly as in Subsection 3.2. The circle β is represented by the equation $\beta(x, y) = 0$ given by (3.1), where by (2.1) we have

$$b = \frac{ca}{(\sqrt{c} - \sqrt{a})^2}. \tag{4.1}$$

Substituting (4.1) in (3.1) and rearranging, we have

$$\beta(x, y) = \frac{a\beta_0 - 2\sqrt{ca}\beta_1 + c\beta_2}{(\sqrt{c} - \sqrt{a})^2},$$

where $\beta_0 = x^2 + y^2$, $\beta_1 = (x + a)^2 + y^2 - a^2$, $\beta_2 = (x + 2a)^2 + (y - a)^2 - a^2$. Since $\beta(x, y) = 0$ implies $a\beta_0 - 2\sqrt{c}\sqrt{a}\beta_1 + c\beta_2 = 0$, $c = 0$ implies $\beta_0 = 0$, $\beta_1 = 0$ and $\beta_2 = 0$ by DBZ, which represent the origin, the circle of radius a and center of coordinates $(-a, 0)$, and the circle of radius a and center of coordinates $(-2a, a)$, respectively. We denote the three figures by β_0, β_1 and β_2, respectively, and indicate in Figure 8 by green.

We consider the circle γ in the case $c = 0$ similarly as in Subsection 3.1. The circle γ has an equation $\gamma(x, y) = (x - 2\sqrt{ca})^2 + (y - c)^2 - c^2 = 0$. Since $\gamma(x, y) = (x^2 + y^2) - 4\sqrt{c}\sqrt{a}x - 2c(y - 2a)$, $c = 0$ implies $x^2 + y^2 = 0$, $x = 0$ and $y - 2a = 0$ by DBZ, which represent the origin, the y-axis and the tangent of α parallel to t. We denote the three figures by γ_0, γ_1 and γ_2, respectively, and indicate in Figure 8 by red.

If the three figures α, γ_i and β_j form a limiting configuration of $\mathcal{W}$, we explicitly denote it by $\mathcal{W}_{ij}$. Then we also get five configurations $\mathcal{W}_{00}$, $\mathcal{W}_{01}$, $\mathcal{W}_{10}$, $\mathcal{W}_{11}$, $\mathcal{W}_{22}$. Since Figure 8 can be obtained if we exchange β_i for γ_i in Figure 2, the configuration $\mathcal{W}_{ij}$ is similar to $\mathcal{W}_{ij}$ in Section 3. Therefore *if we fix the circle α and the line t and consider the limiting configurations of $\mathcal{W}$ in the case $c = 0$, then we have the five limiting configurations, which are the same as the five limiting* *configurations in Section 3, and each of which satisfies exactly one of* (2.1) *and* (2.2).

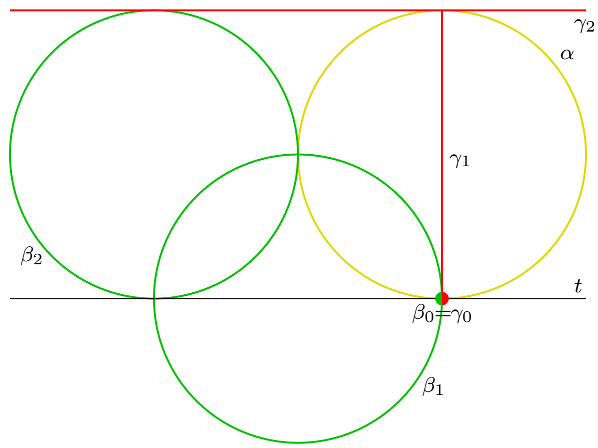

Figure 8: β and γ in the case $c = 0$.

5. Case 3

In this section, we fix the circle γ and the line t, and consider the limiting configurations of $\mathcal{W}$ in the case $b = 0$ using a rectangular coordinates system with origin at the point of tangency of γ and t so that the center of β has coordinates $(2\sqrt{bc}, b)$. The circle β has an equation

$$\beta(x, y) = \left(x - 2\sqrt{bc}\right)^2 + (y - b)^2 - b^2 = 0.$$

Replacing a with c in (3.1), we get the same equation. Hence if $b = 0$, we get $x^2 + y^2 = 0$, $x = 0$, $y = 2c$ by DBZ by a similar consideration made in Subsection 3.1. The last three equations represent the origin, the y-axis, and the tangent of γ parallel to t. We denoted the three figures by β_0, β_1 and β_2, respectively, and indicate in Figure 9 by green.

The circle α has an equation

$$\alpha(x, y) = (x + 2\sqrt{ca})^2 + (y - a)^2 - a^2 = 0, \tag{5.1}$$

while by (2.1), we have

$$a = \frac{bc}{\left(\sqrt{b} - \sqrt{c}\right)^2}. \tag{5.2}$$

Substituting (5.2) in (5.1) and rearranging, we have

$$\alpha(x, y) = \frac{c\alpha_0 - 2\sqrt{bc}\alpha_1 + b\alpha_2}{(\sqrt{b} - \sqrt{c})^2},$$

where $\alpha_0 = x^2 + y^2$, $\alpha_1 = (x - c)^2 + y^2 - c^2$ and $\alpha_2 = (x - 2c)^2 + (y - c)^2 - c^2$. Since $\alpha(x, y) = 0$ implies $c\alpha_0 - 2\sqrt{b}\sqrt{c}\alpha_1 + b\alpha_2 = 0$, $b = 0$ implies $\alpha_0 = 0$, $\alpha_1 = 0$, $\alpha_2 = 0$ by a similar consideration made in Subsection 3.2. The three equation represent the origin and the circle of radius c and center of coordinates $(c, 0)$, and the circle of radius c and center of coordinates $(2c, c)$. We denote the three figures by α_0, α_1, and α_2, respectively, and indicate in Figure 9 by yellow.

If the three figures β_i, α_j and γ form a limiting configuration of $\mathcal{W}$, we explicitly denote it by $\mathcal{W}_{ij}$. Then we have five limiting configurations $\mathcal{W}_{00}$, $\mathcal{W}_{01}$, $\mathcal{W}_{10}$, $\mathcal{W}_{11}$, $\mathcal{W}_{22}$. Replacing α with γ and replacing γ_i with α_i in Figure 2, we get Figure 9. Hence the configuration $\mathcal{W}_{ij}$ is similar to $\mathcal{W}_{ij}$ in Section 3. Therefore *we get five limiting configurations in this case, which are the same as the five limiting configuration in Section 3.*

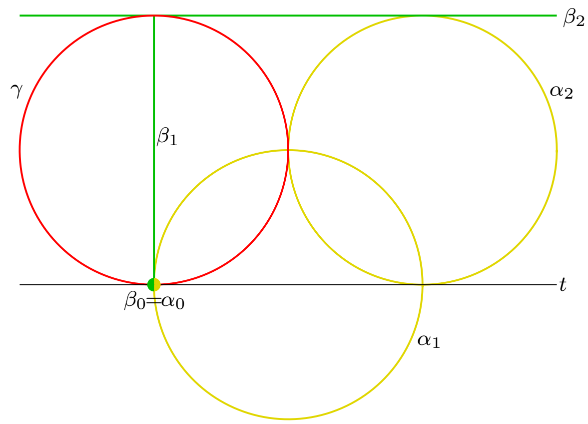

Figure 9: α and β in the case $b = 0$.

6. Conclusion

We have considered the limiting cases of the configuration $\mathcal{W}$ by fixing one of the three circles. The result in this paper shows that the limiting case of each of the six cases consists of five limiting configurations and the five configurations are the same to the configurations $\mathcal{W}_{00}$, $\mathcal{W}_{01}$, $\mathcal{W}_{10}$, $\mathcal{W}_{11}$, $\mathcal{W}_{22}$ in Section 3. In this sense we can say that *the configuration $\mathcal{W}$ has five limiting configurations if one of the circles has radius* 0.

If the three figures forming $\mathcal{W}$ are circles of non-zero radii, then the configuration is uniquely determined, and (2.1) and (2.2) are equivalent and true. Therefore we can consider that (2.1) and (2.2) are essentially the same. On the other hand, if $b = 0$ or $c = 0$, then we can get five configuration, and exactly one of (2.1) and (2.2) is true for each of the configurations. Therefore (2.1) and (2.2) represent different relations in this case.

DBZ $1/0 = 0$ enables us to consider singular cases. The results in this paper shows that singular cases are more interesting with various cases than the usual one. As we have considered in this paper, DBZ has an effect on considering some metric relationships involving radii of circles, which were major topics in Wasan geometry. Applications to Wasan geometry of DBZ and its generalization called *division by zero calculus* can be found in [4], [5], [6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20], [21, 22]. For an extensive reference of the two new concepts, both have been founded by Saburou Saitoh, see [2].

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