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ON MULTIPLE EIGENFUNCTION EXPANSION OF AN OPERATOR PENCIL WITH COMPLEX ALMOST PERIODIC POTENTIALS

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Abstract. In the paper, the multiple eigenfunction expansion of the operator pencil on a whole axis is considered. The coefficients of the considered operators are taken as complex and almost periodic. The eigenfunction expansion of the resolvent is obtained in terms of continuous spectrum eigenfunction and the multiple expansion of arbitrary test functions.

1. INTRODUCTION

In the paper the nonselfadjoint operator pencil L is considered with complex almost periodic potentials, in the space$L_2(-\infty,\infty)$, generated by formal differential expression

$$l\left(\frac{d}{dx},\lambda\right)\equiv\frac{1}{\rho\left(x\right)}\left(-\frac{d^2}{dx^2}+2\lambda p\left(x\right)+q\left(x\right)\right), \tag{1.1}$$

where

$$p\left(x\right)=\sum_{n=1}^{\infty}p_ne^{i\alpha_nx}\ ;\ \ \sum_{n=1}^{\infty}\alpha_n\left|p_n\right|<\infty, \tag{1.2}$$

$$q\left(x\right)=\sum_{n=1}^{\infty}q_ne^{i\alpha_nx}\ ;\ \ \sum_{n=1}^{\infty}\left|q_n\right|<\infty, \tag{1.3}$$

$$\rho\left(x\right)=\begin{cases}1 & x\geq 0\\ -1 & x<0\end{cases}. \tag{1.4}$$

Note that the set $G=\{\alpha_n\}$ satisfied the following conditions

1. $\alpha_1<\alpha_2<.....<\alpha_n<...,\quad \alpha_n\to\infty$.
2. For any $\alpha_i,\alpha_j\in G$ is valid $\alpha_i+\alpha_j\in G$.

Note that many authors have intensively studied the expansion problems in different formulations [1,2,14,15,16,18,19] . This problem for the periodic complex, almost periodic potentials are considered for the first time and obtained here results generalize some previously established ones. Similar problems are considered for the Sturm–Liouville operator in different statements [14,15,16].

In [20] certain non-self-adjoint singular differential operators of even order on a whole axis are investigated, the resolvent is constructed and a multiple spectral expansion problem corresponding to such operators is solved. The scattering

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problem for the Sturm-Liouville operator on the positive half-line with boundary conditions depending quadratically on the spectral parameters is considered in [2]. In [15] the Sturm - Liouville problem under boundary conditions depending on the spectral parameter is considered on the semi-axis. Using the Titchmarsh method, expansion is obtained according to eigenfunctions and the expansion formula is expressed with the scattering data. In [20] is shown that any absolutely continuous function can be expanded in terms of the eigenfunctions of the Sturrn-Liouville problem, where the eigenfunctions have two symmetrically located discontinuities satisfying symmetric jump conditions.

2. Representation of the Solutions

In [7] is proved that the equation

$$-y'' + 2\lambda p(x)y + q(x)y = \lambda^2 \rho(x)y \tag{2.1}$$

has particular solutions of the form

1. *For* $\pm \operatorname{Im}\lambda > 0$ *and* $x > 0$

$$f_1^{\pm}(x,\lambda) = e^{\pm i\lambda x}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm} e^{i\alpha_n x} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n \pm 2\lambda} \sum_{s=n}^{\infty} V_{ns}^{\pm} e^{i\alpha_s x}\right). \tag{2.2}$$

2. *For* $\pm \operatorname{Re}\lambda > 0,$ *and* $x < 0$

$$f_2^{\pm}(x,\lambda) = e^{\pm \lambda x}\left(1 + \sum_{n=1}^{\infty} V_n^{\pm} e^{i\alpha_n x} + \sum_{n=1}^{\infty} \frac{1}{\alpha_n \mp 2i\lambda} \sum_{s=n}^{\infty} V_{ns}^{\pm} e^{i\alpha_s x}\right). \tag{2.3}$$

where the numbers $V_n^{\pm}$, $V_{nn}^{\pm}$ $n < \alpha,\ n, \alpha \in N$ are determined by some recurrent relations. The functions $f_1^{+}(x,\lambda), f_1^{-}(x,\lambda)$ and $f_2^{+}(x,\lambda), f_2^{-}(x,\lambda)$ are linearly independent for $\lambda \neq 0,\ \lambda \neq \mp\frac{\alpha_n}{2}, \lambda_n = \mp\frac{i\alpha_n}{2}$ and

$$f_{n1}^{\pm}(x) = \lim_{\lambda \to \mp\frac{\alpha_n}{2}} (\alpha_n \pm 2\lambda) f_1^{\pm}(x,\lambda) = V_{nn}^{\pm} f_1^{\mp}\left(x, \mp\frac{\alpha_n}{2}\right), \tag{2.4}$$

$$f_{n2}^{\pm}(x) = \lim_{\lambda \to \mp i\frac{\alpha_n}{2}} (\alpha_n \mp 2i\lambda) f_2^{\pm}(x,\lambda) = V_{nn}^{\pm} f_2^{\mp}\left(x, \mp i\frac{\alpha_n}{2}\right). \tag{2.5}$$

The functions $f_1^{\pm}(x,\lambda), f_2^{\pm}(x,\lambda)$ can be extended as a solution of equation (2.1) respectively as follows

$$f_2^{+}(x,\lambda) = A(\lambda) f_1^{+}(x,\lambda) + C(\lambda) f_1^{-}(x,\lambda),\ \ \lambda > 0, \operatorname{Im}\lambda = 0, \tag{2.6}$$

$$f_2^{-}(x,\lambda) = B(\lambda) f_1^{+}(x,\lambda) + D(\lambda) f_1^{-}(x,\lambda),\ \ \lambda < 0, \operatorname{Im}\lambda = 0, \tag{2.7}$$

$$f_1^{+}(x,\lambda) = iD(\lambda) f_2^{+}(x,\lambda) - iC(\lambda) f_2^{-}(x,\lambda),\ \ Im\lambda > 0, \operatorname{Re}\lambda = 0, \tag{2.8}$$

$$f_1^{-}(x,\lambda) = -iB(\lambda) f_2^{+}(x,\lambda) + iA(\lambda) f_2^{-}(x,\lambda),\ \ Im\lambda > 0, \operatorname{Re}\lambda = 0, \tag{2.9}$$

where

$$A(\lambda) = \frac{1}{2i\lambda} W[f_2^+(x,\lambda), f_1^-(x,\lambda)],\ \lambda \in S_3, \tag{2.10}$$

$$B(\lambda) = \frac{1}{2i\lambda} W[f_2^-(x,\lambda), f_1^-(x,\lambda)], \lambda \in S_2, \tag{2.11}$$

$$C(\lambda) = \frac{1}{2i\lambda} W[f_1^+(x,\lambda), f_2^+(x,\lambda)],\ \lambda \in S_0, \tag{2.12}$$

$$D(\lambda) = \frac{1}{2i\lambda} W[f_1^+(x,\lambda), f_2^-(x,\lambda)],\ \lambda \in S_1, \tag{2.13}$$

if to denote by $W[f,g] = f'g - fg'$.

Here $S_k = \left\{ \frac{k\pi}{2} < \arg\lambda < \frac{(k+1)\pi}{2} \right\},\ k = \overline{0,3}$ are the sectors of the complex λ-plane.

It is proved [7] that the resolvent of the operator pencil L has a form

$$[R_0(x,t,\lambda) = -\frac{1}{2i\lambda C(\lambda)} \begin{cases} f_1^+(x,\lambda) f_2^+(t,\lambda), & t < x \\ f_1^+(t,\lambda) f_2^+(x,\lambda), & t > x \end{cases};\ \lambda \in S_0, \tag{2.14}$$

$$R_1(x,t,\lambda) = -\frac{1}{2i\lambda D(\lambda)} \begin{cases} f_1^+(x,\lambda) f_2^-(t,\lambda), & t < x \\ f_1^+(t,\lambda) f_2^-(x,\lambda), & t > x \end{cases};\ \lambda \in S_1, \tag{2.15}$$

$$R_2(x,t,\lambda) = \frac{1}{2i\lambda B(\lambda)} \begin{cases} f_1^-(x,\lambda) f_2^-(t,\lambda), & t < x \\ f_1^-(t,\lambda) f_2^-(x,\lambda), & t > x \end{cases};\ \lambda \in S_2\ , \tag{2.16}$$

$$R_3(x,t,\lambda) = \frac{1}{2i\lambda A(\lambda)} \begin{cases} f_1^-(x,\lambda) f_2^+(t,\lambda), & t < x \\ f_1^-(t,\lambda) f_2^+(x,\lambda), & t > x \end{cases};\ \lambda \in S_3. \tag{2.17}$$

The spectra of the operator pencil L has no pure real and imaginary eigenvalues, the residual spectrum is empty and the continuous spectrum consists of axes $\{Re\lambda = 0\} \cup \{Im\lambda = 0\}$ and may have a spectral singularity at the points $\lambda = \pm\frac{\alpha_n}{2}$, $\lambda = \pm\frac{i\alpha_n}{2}$, $n \in N$. The eigenvalues of L are finite and coincide with the zeros of the functions $A(\lambda), B(\lambda), C(\lambda), D(\lambda)$ from the sectors $S_k,\ k = \overline{0,3}$.

3. Eigenfunction expansion of resolvent in terms of continuous spectrum eigenfunction

Let us rewrite the operator pencil L generated by the differential expression (1.1) as

$$L(\lambda) = -\rho(x)\lambda^2 + A_1\lambda + A_0.$$

Here $\rho(x)$ is defined as (1.4) and

$$A_1 = 2p(x),\ \ A_0 = q(x) - \frac{d^2}{dx^2}.$$

Theorem 3.1. *Resolvent operator $R(x,t,\lambda)$ of the operator pencil L admits expansion in terms of continuous spectrum eigenfunction*

$$\begin{array}{l} R(x,t,z) = -\sum\limits_{i=1}^{n} res\left(R(x,t,\lambda)\right)|_{\lambda=\lambda_i} + \int\limits_{\Gamma_0^-} \frac{f_2^+(x,\lambda) f_2^+(t,\lambda)}{2i\lambda(\lambda-z)A(\lambda)C(\lambda)} d\lambda + \\ + \int\limits_{\Gamma_1^-} \frac{f_1^+(x,\lambda) f_1^+(t,\lambda)}{2i\lambda(\lambda-z)D(\lambda)C(\lambda)} d\lambda + \int\limits_{\Gamma_2^-} \frac{f_2^-(x,\lambda) f_2^-(t,\lambda)}{2i\lambda(\lambda-z)D(\lambda)B(\lambda)} d\lambda + \\ + \int\limits_{\Gamma_3^-} \frac{f_1^-(x,\lambda) f_1^-(t,\lambda)}{2i\lambda(\lambda-z)A(\lambda)B(\lambda)} d\lambda + \frac{2}{i\left(\frac{\alpha_n}{2}-z\right)\alpha_n} V_{nn}^- f_1^+\left(x,\frac{\alpha_n}{2}\right) f_1^+\left(t,\frac{\alpha_n}{2}\right) + \\ + \frac{2i}{\left(-\frac{\alpha_n}{2}-z\right)\alpha_n} V_{nn}^+ f_1^-\left(x,-\frac{\alpha_n}{2}\right) f_1^-\left(t,-\frac{\alpha_n}{2}\right). \end{array} \tag{3.1}$$

Proof. By the definition of the resolvent operator, we can write

$$L(\lambda) R(x,t,\lambda) = E$$

or

$$\left(-\rho(x)\lambda^2 + A_1\lambda + A_0\right) R(x,t,\lambda) = E.$$

Then we have

$$-\rho(x)\lambda^2 R(x,t,\lambda) = E - (A_1\lambda + A_0) R(x,t,\lambda).$$

From this, it can be easily obtained the following relation

$$\begin{array}{l} R(x,t,\lambda) = -\frac{E}{\rho(x)\lambda^2} - \frac{E}{\rho(x)\lambda^2}(A_1\lambda + A_0) R(x,t,\lambda) = \\ = -\frac{E}{\rho(x)\lambda^2} - \frac{E}{\rho(x)\lambda^2}(A_1\lambda + A_0)\left[-\frac{E}{\rho(x)\lambda^2} - \frac{E}{\rho(x)\lambda^2}(A_1\lambda + A_0) R(x,t,\lambda)\right] = \\ = -\frac{E}{\rho(x)\lambda^2} + \frac{E}{\lambda^4}(A_1\lambda + A_0) - \frac{E}{\lambda^4}(A_1\lambda + A_0)^2 R(x,t,\lambda). \end{array}$$

Let $\Gamma_0^+\left(\Gamma_0^-\right)$ be a contour formed by segments $[0,\frac{\alpha_1}{2}-\delta]$, $[\frac{\alpha_2}{2}+\delta,\frac{\alpha_3}{2}-\delta]$, ..., $[\frac{\alpha_n}{2}+\delta,\frac{\alpha_{n+1}}{2}-\delta]$ and semicircles of the radius δ with centers at points $\frac{\alpha_n}{2}, n \in N$ located in the upper (lower) half-plane.

Let $\Gamma_\nu^+\left(\Gamma_\nu^-\right)$ be obtained from $\Gamma_0^+\left(\Gamma_0^-\right)$ by rotation of an angle $\frac{\nu\pi}{2}$, $\nu = 1,2,3$ and the numbers $\lambda_1, \lambda_2, \lambda_3 .... \lambda_n$ be an eigenvalue of the operator $L(\lambda)$.

Then we can choose a sufficiently large number Nso that all numbers λ_1, λ_2, $\lambda_3 .... \lambda_n$ will lie inside the circle with the radius.

For that case we have

$$\begin{array}{l}\frac{1}{2i\pi}\int\limits_{\Gamma_N}\frac{R(x,t,\lambda)}{\lambda-z}d\lambda=R(x,t,z)+\sum\limits_{i=1}^{n}res\left(R(x,t,\lambda)\right)|_{\lambda=\lambda_i}+\\+\sum\limits_{\nu=0}^{3}\int\limits_{\Gamma_\nu^-}\frac{R_{\nu-1}(x,t,\lambda)}{\lambda-z}d\lambda+\int\limits_{\Gamma_\nu^+}\frac{R_\nu(x,t,\lambda)}{\lambda-z}d\lambda=\\=R(x,t,z)+\sum\limits_{i=1}^{n}res\left(\frac{R(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=\lambda_i}+\\+\sum\limits_{\nu=0}^{3}\int\limits_{\Gamma_\nu^-}\frac{R_{\nu-1}(x,t,\lambda)-R_\nu(x,t,\lambda)}{\lambda-z}d\lambda+\\res\left(\frac{R_0(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=\frac{\alpha_n}{2}}+res\left(\frac{R_3(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=-i\frac{\alpha_n}{2}}+\\+res\left(\frac{R_2(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=-\frac{\alpha_n}{2}}+res\left(\frac{R_1(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=i\frac{\alpha_n}{2}}.\end{array}\tag{3.2}$$

Here we denoted $R_{-1}(x,t,\lambda)=R_3(x,t,\lambda)$. Then taking into account that

$$\begin{array}{l}|R(x,t,\lambda)|\le\frac{C}{|\lambda|}e^{-\tau|x-t|}\\C=C(\lambda),\tau=\min\{\mathrm{Im}\lambda,\mathrm{Re}\lambda\},\forall x,t\in R\end{array}\tag{3.3}$$

for $N\to\infty$. Thus for $|\lambda|\to\infty$ we have

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N}\frac{R(x,t,\lambda)}{\lambda-z}d\lambda=0.$$

Then relation (3.2) can be rewritten as follows

$$\begin{array}{l}R(x,t,z)=-\sum\limits_{i=1}^{n}res\left(\frac{R(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=\lambda_i}-\\-\sum\limits_{\nu=0}^{3}\int\limits_{\Gamma_\nu^-}\frac{R_{\nu-1}(x,t,\lambda)-R_\nu(x,t,\lambda)}{\lambda-z}d\lambda-\\-res\left(\frac{R_0(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=\frac{\alpha_n}{2}}-res\left(\frac{R_3(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=-i\frac{\alpha_n}{2}}-\\-res\left(\frac{R_2(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=-\frac{\alpha_n}{2}}-res\left(\frac{R_1(x,t,\lambda)}{\lambda-z}\right)\Big|_{\lambda=i\frac{\alpha_n}{2}}.\end{array}\tag{3.4}$$

By using formulas (2.6-2.9) and (2.14-2.17) we have

$$\begin{array}{l}R_0(x,t,\lambda)-R_3(x,t,\lambda)=\\=\frac{1}{2i\lambda C(\lambda)}f_1^+(x,\lambda)f_2^+(t,\lambda)-\frac{1}{2i\lambda A(\lambda)}f_1^-(x,\lambda)f_2^+(t,\lambda)=\\=\frac{1}{2i\lambda A(\lambda)C(\lambda)}\left[A(\lambda)f_1^+(x,\lambda)f_2^+(t,\lambda)-C(\lambda)f_1^-(x,\lambda)f_2^+(t,\lambda)\right]=\\=\frac{1}{2i\lambda A(\lambda)C(\lambda)}\{-\frac{1}{2i\lambda}[f_2^+(x,\lambda)f_1^{-'}(x,\lambda)-\\-f_2^{+'}(x,\lambda)f_1^-(x,\lambda)]f_1^+(x,\lambda)f_2^+(t,\lambda)+\\+\frac{1}{2i\lambda}[f_2^+(x,\lambda)f_1^{+'}(x,\lambda)-f_2^{+'}(x,\lambda)f_1^+(x,\lambda)]f_1^+(x,\lambda)f_2^+(t,\lambda)\}=\\=\frac{1}{2i\lambda A(\lambda)C(\lambda)}\{\frac{1}{2i\lambda}f_2^+(x,\lambda)f_2^+(t,\lambda)[f_1^{+'}(x,\lambda)f_1^+(x,\lambda)-\\-f_1^{-'}(x,\lambda)f_1^+(x,\lambda)]\}=\frac{1}{2i\lambda A(\lambda)C(\lambda)}f_2^+(x,\lambda)f_2^+(t,\lambda).\end{array}$$

In the same way, we obtain

$$\begin{array}{l} R_1(x,t,\lambda) - R_0(x,t,\lambda) = \frac{1}{2i\lambda D(\lambda)C(\lambda)} f_1^+(x,\lambda) f_1^+(t,\lambda), \\ R_2(x,t,\lambda) - R_1(x,t,\lambda) = \frac{1}{2i\lambda D(\lambda)B(\lambda)} f_2^-(x,\lambda) f_2^-(t,\lambda), \\ R_3(x,t,\lambda) - R_2(x,t,\lambda) = \frac{1}{2i\lambda A(\lambda)B(\lambda)} f_1^-(x,\lambda) f_1^-(t,\lambda). \end{array}$$

Now we calculate the residual terms

$$\begin{array}{l} \lim\limits_{\lambda\to\frac{\alpha_n}{2}} (\alpha_n - 2\lambda) R_0(x,t,\lambda) = \\ \lim\limits_{\lambda\to\frac{\alpha_n}{2}} (\alpha_n - 2\lambda) \frac{1}{2i\lambda} f_1^+(x,\lambda) \left[\frac{A(\lambda)}{C(\lambda)} f_1^+(t,\lambda) + f_1^-(t,\lambda)\right] = \\ = \frac{2}{i\alpha_n} V_{nn}^- f_1^+\left(x,\frac{\alpha_n}{2}\right) f_1^+\left(t,\frac{\alpha_n}{2}\right), \end{array} \tag{3.5}$$

$$\lim_{\lambda\to-\frac{\alpha_n}{2}} (\alpha_n + 2\lambda) R_2(x,t,\lambda) = \frac{2i}{\alpha_n} V_{nn}^+ f_1^-\left(x,-\frac{\alpha_n}{2}\right) f_1^-\left(t,-\frac{\alpha_n}{2}\right), \tag{3.6}$$

$$\begin{array}{l} \lim\limits_{\lambda\to-\frac{i\alpha_n}{2}} (\alpha_n - 2i\lambda) R_3(x,t,\lambda) = \\ = \lim\limits_{\lambda\to-\frac{i\alpha_n}{2}} (\alpha_n - 2i\lambda) \frac{1}{2\lambda} \left[-\frac{B(\lambda)}{A(\lambda)} f_2^+(x,\lambda) + f_2^-(x,\lambda)\right] f_2^+(t,\lambda) = 0, \end{array} \tag{3.7}$$

$$\begin{array}{l} \lim\limits_{\lambda\to\frac{i\alpha_n}{2}} (\alpha_n + 2i\lambda) R_1(x,t,\lambda) = \\ \lim\limits_{\lambda\to\frac{i\alpha_n}{2}} (\alpha_n + 2i\lambda) \frac{1}{2\lambda} \left[-i f_2^+(x,\lambda) + i\frac{C(\lambda)}{D(\lambda)} f_2^-(x,\lambda)\right] f_2^-(t,\lambda) = 0. \end{array} \tag{3.8}$$

The theorem will be proved by substitution (3.5)-(3.8) into relation (3.4). The theorem is proved.

□

4. Multiple eigenfunction expansion

Let $f_i(x)$, $i = 0,1$ be arbitrary functions that are identically equal to zero in some neighbourhood of infinite and 4 times differentiable. Consider the differential equation

$$-y'' + (2\lambda p(x) + q(x))y - \lambda^2 \rho(x) y = \lambda\varphi_0(x) + \varphi_1(x), \tag{4.1}$$

where

$$\varphi_0(x) = f_0(x), \quad \varphi_1(x) = f_1(x) + A_1(x)\rho(x) f_0(x).$$

Using the general theory of differential equations, the solution of equation (4.1) can be written as

$$y(x,\lambda) = R(x,t,\lambda)(\lambda\varphi_0(x) + \varphi_1(x)). \tag{4.2}$$

Theorem 4.1. *For the sufficiently large* $\lambda \in \bigcup\limits_{\nu=0}^{3} S_\nu$ *we have the representation*

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N} \lambda^m y\left(x,\lambda\right) d\lambda = -f_m + o\left(\varepsilon\right) + O\left(\frac{1}{\left|\lambda\right|^2}\right),\ m = 0, 1.$$

Proof. Let us denote by Γ_N the circle centred at the origin with the radius $N$and by $\Gamma_{N,\varepsilon}$ the circle centred at the origin with the radius N from which removed a finite number of arcs with total length ε. Then well known that [9]

$$\int\limits_{\Gamma_N} \lambda^m d\lambda = \begin{cases} 0 & m \neq -1 \\ 2i\pi & m = -1 \end{cases} \tag{4.3}$$

and

$$\left|\int\limits_{\Gamma_{N,\varepsilon}} \lambda^m d\lambda\right| \leq \left\{ \begin{array}{cc} \varepsilon R^{m+1} & m \neq -1 \\ 2\pi - \varepsilon & m = 1 \end{array} \right. . \tag{4.4}$$

Now by taking into account (4.3), (4.4) and

$$y\left(x,\lambda\right) = R\left(x,t,\lambda\right) f\left(t\right) = \int\limits_{-\infty}^{\infty} R\left(x,\xi,\lambda\right) f\left(\xi\right) d\xi$$

we have

$$\begin{aligned}
&\tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} y\left(x,\lambda\right) d\lambda = \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} R\left(x,t,\lambda\right)\left(\lambda\varphi_0\left(x\right) + \varphi_1\left(x\right)\right) d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} [-\tfrac{E}{\rho(x)\lambda^2} + \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right) + \\
&+ \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 R\left(x,t,\lambda\right)](\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\left(x\right)\rho\left(x\right) f_0\left(x\right)\right)) d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} -\tfrac{f_0(x)}{\rho(x)\lambda} d\lambda + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)) d\lambda + \\
&+ \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 R\left(x,t,\lambda\right)\left(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)\right) d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} -\tfrac{f_0(x)}{\rho(x)\lambda} d\lambda + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)) d\lambda + \\
&+ \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 \int\limits_{-\infty}^{\infty} R\left(x,\xi,\lambda\right)\left(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)\right) d\xi d\lambda = \\
&= -\tfrac{f_0(x)}{\rho(x)} + o\left(\varepsilon\right) + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{M_1}{\lambda^4}\left(A_1\lambda + A_0\right)^2 d\lambda
\end{aligned}$$

Here

$$M_1 = \int\limits_{-\infty}^{\infty} R\left(x,\xi,\lambda\right)\left(\lambda f_1\left(\xi\right) + f_0\left(\xi\right)\right) d\xi,\ \ \lambda \in \bigcup\limits_{\nu=0}^{3} S_\nu.$$

Taking into account (3.3) for sufficiently large $\lambda \in \bigcup\limits_{\nu=0}^{3} S_\nu$ we have

$$\int\limits_{\Gamma_{N,\varepsilon}} \frac{M_1}{\lambda^4}(A_1\lambda + A_0)^2 d\lambda = O\left(\frac{1}{|\lambda^2|}\right).$$

Thus we finally have for sufficiently large $\lambda \in \bigcup\limits_{\nu=0}^{3} S_\nu$

$$\frac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} y\left(x,\lambda\right)d\lambda = -\frac{f_0\left(x\right)}{\rho\left(x\right)} + o\left(\varepsilon\right) + O\left(\frac{1}{|\lambda^2|}\right)$$

where $|\lambda| \to \infty$ the right side is

$$-\frac{f_0\left(x\right)}{\rho\left(x\right)}.$$

In the same way, we have for the sufficiently large $\lambda \in \bigcup\limits_{\nu=0}^{3} S_\nu$ we obtain

$$\begin{aligned}
&\tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \lambda y\left(x,\lambda\right)d\lambda = \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} R\left(x,t,\lambda\right)\left(\lambda^2\varphi_0\left(x\right) + \lambda\varphi_1\left(x\right)\right)d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} [-\tfrac{E}{\rho(x)\lambda^2} + \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right) + \\
&+\tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 R\left(x,t,\lambda\right)](\lambda^2 f_0\left(x\right) + \lambda(f_1\left(x\right) + A_1\left(x\right)\rho\left(x\right)f_0\left(x\right))d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} -\tfrac{f_1(x)+A_1\rho(x)f_0(x)}{\rho(x)\lambda}d\lambda + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{A_1 f_0(x)}{\lambda}d\lambda + \\
&+\tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)(\lambda^2 f_0\left(x\right) + \lambda f_1\left(x\right))d\lambda + \\
&+\tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 R\left(x,t,\lambda\right)\left(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)\right)d\lambda = \\
&= \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} -\tfrac{f_1(x)}{\rho(x)\lambda}d\lambda + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right))d\lambda + \\
&+\tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{E}{\lambda^4}\left(A_1\lambda + A_0\right)^2 \int\limits_{-\infty}^{\infty} R\left(x,\xi,\lambda\right)\left(\lambda f_0\left(x\right) + \left(f_1\left(x\right) + A_1\rho f_0\right)\right)d\xi d\lambda = \\
&= -\tfrac{f_1(x)}{\rho(x)} + o\left(\varepsilon\right) + \tfrac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \tfrac{M_1}{\lambda^4}\left(A_1\lambda + A_0\right)^2 d\lambda = -\tfrac{f_1(x)}{\rho(x)} + o\left(\varepsilon\right) + O(\tfrac{1}{\lambda^2}).
\end{aligned}$$

Thus we showed that

$$\frac{1}{2i\pi}\int\limits_{\Gamma_{N,\varepsilon}} \lambda^m y\left(x,\lambda\right)d\lambda = -\rho\left(x\right)f_m\left(x\right) + o\left(\varepsilon\right) + O(\frac{1}{\lambda^2}),\ m = 0,1. \tag{4.5}$$

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N} \lambda^m y\left(x,\lambda\right)d\lambda = -\rho\left(x\right)f_m\left(x\right) + O(\frac{1}{\lambda^2}),\ m = 0,1. \tag{4.6}$$

For (4.5) at $|\lambda| \to \infty$ we have

$$
\begin{aligned}
&\lim_{N\to\infty}\tfrac{1}{2i\pi}\int\limits_{\Gamma_N}\lambda^m y\,(x,\lambda)d\lambda=\tfrac{1}{2i\pi}\sum_{\nu=0}^{3}\Big[\int\limits_{\Gamma_\nu^+}\lambda^m y\,(x,\lambda)\,d\lambda+\int\limits_{\Gamma_\nu^-}\lambda^m y\,(x,\lambda)\,d\lambda\Big]=\\
&=\tfrac{1}{2i\pi}\sum_{\nu=0}^{3}\Big\{\int\limits_{\Gamma_\nu^+}\lambda^m\int\limits_{-\infty}^{\infty}R_\nu\,(x,\xi,\lambda)\,[\lambda\varphi_0\,(x)+\varphi_1\,(x)]+\\
&+\int\limits_{\Gamma_\nu^-}\lambda^m\int\limits_{-\infty}^{\infty}R_{\nu-1}\,(x,\xi,\lambda)\,[\lambda\varphi_0\,(x)+\varphi_1\,(x)]\Big\}d\xi d\lambda=\\
&=\tfrac{1}{2i\pi}\sum_{\nu=0}^{3}\Big\{\int\limits_{\Gamma_\nu^+}\lambda^m\int\limits_{-\infty}^{\infty}[R_\nu\,(x,\xi,\lambda)-R_{\nu-1}\,(x,\xi,\lambda)]\,[\lambda\varphi_0\,(x)+\varphi_1\,(x)]\,d\xi d\lambda+\\
&+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty}\operatorname*{Res}_{\lambda=\frac{\alpha_n}{2}}R_0\,(x,\xi,\lambda)\left[\tfrac{\alpha_n}{2}\varphi_0\,(x)+\varphi_1\,(x)\right]d\xi+\\
&+\sum_{n=1}^{\infty}\int\limits_{-\infty}^{\infty}\operatorname*{Res}_{\lambda=-\frac{\alpha_n}{2}}R_2\,(x,\xi,\lambda)\left[-\tfrac{\alpha_n}{2}\varphi_0\,(x)+\varphi_1\,(x)\right]d\xi.
\end{aligned}
\tag{4.7}
$$

if we will take into account the fact that for sufficiently large λ

$$\frac{1}{2i\pi}\int\limits_{\Gamma_N}\lambda^m y\,(x,\lambda)d\lambda=-f_m+o\,(\varepsilon)+O\left(\frac{1}{|\lambda|^2}\right). \tag{4.8}$$

then the last formulae (4.8) will be a multiple eigenvalue expansion by the Eigenfunction of the continuous spectrum.

□

5. CONCLUSION.

In this work, the expansion formula is obtained according to the eigenfunctions for the non-self-adjoint operators with complex and almost periodic potential. In this study, the expansion formula is obtained in terms of continuous spectrum eigenfunction and written down the multiple expansion of arbitrary test functions. The techniques and methods can be applied to the other different equations with different boundary conditions and new expansion formulas can be obtained.

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