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HARMONIC NEUMANN FUNCTION FOR A PLANAR CIRCULAR RECTANGLE - A CASE STUDY

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Abstract. Recently an application of the parqueting-reflection principle led to the harmonic Green function for a circular rectangle in the complex plane and to the Poisson kernel function. Knowing the harmonic Green function then also the Bergman kernel function is determined. The same procedure as for constructing the Green function produces the harmonic Neumann function explicitly. It is at the same time the Neumann function for any of the countably many domains of the parqueting of the complex plane from the reflection process starting from the original circular rectangle. This is worked out here. The Neumann function constructed is in so far particular as it fails to be symmetric in its variables due to the necessity of inserting convergence generating factors in the infinite product determining the function. This results in different expressions for the piecewise continuous normal derivatives of the function with respect to the $z-$ and the $\zeta-$ variable.

1. Basics

The harmonic Neumann function for a domain D of the complex plane $\mathbb{C}$ is a fundamental solution to the Laplace equation adjusted to the Neumann boundary condition. It is real-valued and does exist for domains D with piecewise smooth boundary ∂D. The Neumann boundary condition prescribes the normal derivative of functions defined in the domain. Here the outward normal derivative on ∂D is $\partial_\nu = \nabla \cdot \nu$, with the gradient differential operator ∇ and the outward unit normal vector ν. It exists on the smooth parts of ∂D. For the Neumann function this normal derivative is piecewise continuous on ∂D.

Naturally the Neumann function is not uniquely defined just by its boundary condition as an additional additive constant will neither disturb the harmonic nature nor violate the boundary condition. Thus a normalization condition is needed for uniqueness. As the harmonic Green function also the Neumann function is conformally invariant. This also has consequences for the Bergman kernel function, see e.g. [1, 2, 3, 4]. Hence, they are all known for those simply connected domains conformally equivalent to the unit disc. A similar situation appears for doubly connected domains where the concentric ring serves as prototype [5]. While for the unit disc the normal derivative of the Neumann function on the unit circle is constantly -2 it is piecewise constant for a concentric ring, namely -2 on the

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outer circle while it is 0 on the inner circle [5]. Other examples are explained e.g. in [6].

Definition (Neumann function). The Neumann function $N_1(z,\zeta)$ for D has for any $\zeta \in D$ the properties:

- $N_1(\,\cdot\,,\zeta)$ is harmonic in $D\setminus\{\zeta\}$ and continuously differentiable in $\overline{D}\setminus\{\zeta\}$ up to the corner points of ∂D,
- $h_2(z,\zeta) = N_1(z,\zeta) + \log|z-\zeta|^2$ is harmonic for $z \in D$,
- $\partial_{\nu_z} N_1(z,\zeta) = \sigma(s)$ for $z = z(s) \in \partial D$, ∂_{ν_z} is the outward normal derivative on the smooth parts of ∂D and s is the arc length parameter of ∂D. The density function σ is a real-valued, piecewise continuous function of s with finite mass $\int\limits_{\partial D} \sigma(s)ds$,
- $\int\limits_{\partial D} \sigma(s_z) N_1(z,\zeta)\, ds_z = 0$ (normalization condition).

Moreover the Neumann function is symmetric, $N_1(z,\zeta) = N_1(\zeta,z)$ for $z,\zeta \in D$, $z \neq \zeta$. For a proof see [7]. All four properties of the Neumann function are contributing to this proof.

Remark 1.1. A problem to determine the Neumann function for a given domain is to find the density function σ. For the unit disk $\sigma(z) = -2$ for $|z| = 1$, for the concentric ring $R = \{0 < r < |z| < 1\}$

$$\sigma(z) = \begin{cases} -2, & |z| = 1, \\ 0, & |z| = r. \end{cases}$$

In fact the normalization condition for the ring R is $\int\limits_{|z|=1} N_1(z,\zeta)\, ds_z = 0$ for $\zeta \in R$.

The density σ is piecewise continuous as a function of $z \in \partial D$ but may depend on ζ. The variable ζ may even belong to the closure $\overline{D}$ as long as it does not lie on the particular part of the boundary. For an example see the hyperbolic strip ([8]) $D = \mathbb{D} \setminus (\overline{D_{-m_1}(r_1)} \cup \overline{D_{m_2}(r_2)}) = \{|z| < 1, r_1 < |z+m_1|, r_2 < |z-m_2|\}$, where the parameters $0 < m_1, m_2, 0 < r_1, r_2, m_1^2 = r_1^2 + 1, m_2^2 = r_2^2 + 1$, for the discs $D_{-m_1}(r_1)$ and $D_{m_2}(r_2)$ are related via $r_1 + r_2 < m_1 + m_2$ so that $-1 < r_1 - m_1 < 0 < m_2 - r_2 < 1$. Here $D_m(r) = \{|z-m| < r\}$.

For any $z \in D$ the normal derivative of $N_1(z,\cdot)$ at the boundary $\partial D \setminus \{-\frac{1}{m_1} \pm i\frac{r_1}{m_1}, \frac{1}{m_2} \pm i\frac{r_2}{m_2}\}$ is

$$\partial_{\nu_\zeta} N_1(z,\zeta) = \begin{cases} -2, \zeta \in \partial D \cap \partial\mathbb{D}, \\ 2 - 4\frac{m_1-1}{|\zeta+1|^2}, \zeta \in \partial D \cap \partial D_{-m_1}(r_1), \\ 2 - 4\frac{m_2-1}{|\zeta-1|^2}, \zeta \in \partial D \cap \partial D_{m_2}(r_2), \end{cases}$$

while the normal derivative with respect to the variable z on ∂D for any $\zeta \in \overline{D}$ but not on the respective boundary arc is

$$\partial_{\nu_z} N_1(z,\zeta) = \begin{cases} -2, z \in \partial D \cap \partial \mathbb{D}, \zeta \in \overline{D} \setminus \partial \mathbb{D}, \\ -\dfrac{1-|z|^2}{|z|^2}, z \in \partial D \cap \partial D_{-m_1}(r_1), \zeta \in \overline{D} \setminus \partial D_{-m_1}(r_1), \\ \qquad\qquad z \in \partial D \cap \partial D_{m_2}(r_2), \zeta \in \overline{D} \setminus \partial D_{m_2}(r_2). \end{cases}$$

This Neumann function for the hyperbolic strip from [9] is obviously not symmetric. The reason is that the normalization condition is not satisfied. If a fundamental solution of the Laplacian does satisfy the first three conditions for the Neumann function an arbitrary harmonic function in the variable ζ may be added without altering these three conditions. This altered function still serves to solve the Neumann boundary value problem for the Poisson equation. However, the solution is only defined up to an arbitrary additive constant. Only the normalization of the Neumann function serves to determine this constant via a proper additional side condition, see [9, 10].

The parqueting-reflection principle is a convenient way to determine the Green and Neumann functions for a class of plane domains the boundary of which are composed from arcs from circles and straight lines such that repeated reflections at the boundary parts provide a parqueting of the complex plane, e.g. [11]. Here even multiply coverings are admissible, see e.g. [12]. But if the covering requires a countable set of reflections then an infinite product of linear factors appear which requires convergence generating factors. They however, seem to violate the symmetry of the Neumann function. Even for the Green function the symmetry is not obvious and its verification needs some effort, see e.g. [8]. The same procedure can be used for determining the Schwarz kernel function [13].

In [14] the Green function is constructed for a certain circular rectangle

$$D = \{0 < i(\overline{z} - z), |z \mp 1| < \sqrt{2} < |z - i\sqrt{3}|\}.$$

It is the upper half of the circular rectangle $\{|z \mp 1| < \sqrt{2} < |z \mp i\sqrt{3}|\}$ bounded by four circles. The reflection process for the domain D exploited in [14] in detail leads to a sequence of discs $|z - im_k| < r_k, m_k^2 = r_k^2 + 1, k \in \mathbb{N}$, with $m_1 = \sqrt{3}, m_{k+1} = \frac{1+m_1 m_k}{m_1 + m_k}$, and to a sequence of basic points $z_k^+, |z_k^+ - im_k| < r_k$ starting from $z \in D$, denoted by z_0^+ and given through

$$z_{k+1}^+ = \frac{im_{k+1}\overline{z_k^+} - 1}{\overline{z_k^+} + im_{k+1}}.$$

and to $z_k^{++} = \frac{1-\overline{z_k^+}}{1+\overline{z_k^+}}$ also lying in $|z - im_k| < r_k$.

If the point z_{2k+1}^+ is on the boundary of the disc $|z - im_{2k+1}| = r_{2k+1}$ then it coincides with its pre-image z_{2k}^+.

Lemma 1.2. *If for* $k \in \mathbb{N}_0, |z_{2k+1}^+ - im_{2k+1}| = r_{2k+1}$*, then* $z_{2k}^+ = z_{2k+1}^+$.

Proof. From the iterative definition of the points z_k^+ follows

$$(z_{2k+1}^+ - im_{2k+1})(\overline{z_{2k}^+} + im_{2k+1}) - r_{2k+1}^2 = 0.$$

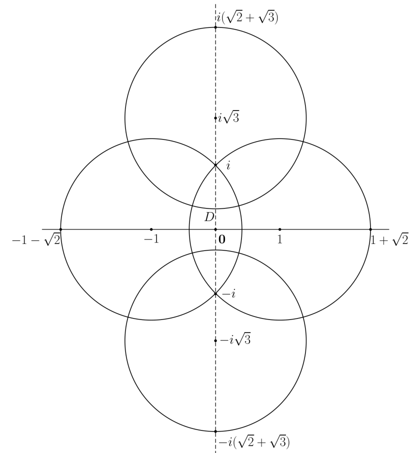

FIGURE 1. Basic Circles

But as the assumption means

$$(z^+_{2k+1} - im_{2k+1})(\overline{z^+_{2k+1}} + im_{2k+1}) - r^2_{2k+1} = 0,$$

then $z^+_{2k} = z^+_{2k+1}$. □

Corollary 1.3. *If for $k \in \mathbb{N}_0, z^+_{2k} = z^+_{2k+1}$ hold, then $z^+_{2k+2} = z^+_{2k+3}$.*

Reflections of these z^+_k-, z^{++}_k-points in the upper right-hand quarter of the complex plane as well at the circle $|z-1| = \sqrt{2}$ as at the real and at the imaginary axes produce a set of points related to the initiated covering of the plane and lead to the Green function for D in the form $G_1(z,\zeta) = \log |P(z,\zeta)|^2$ with

$$P(z,\zeta) = \prod_{k\in\mathbb{N}_0} \frac{\zeta - z^+_{2k+1}}{\zeta - z^+_{2k}} \frac{\zeta - \overline{z^+_{2k}}}{\zeta - \overline{z^+_{2k+1}}} \frac{\zeta - z^{++}_{2k}}{\zeta - z^{++}_{2k+1}} \frac{\zeta - \overline{z^{++}_{2k+1}}}{\zeta - \overline{z^{++}_{2k}}}$$
$$\times \frac{1 + \overline{z^+_{2k}}\zeta}{1 + \overline{z^+_{2k+1}}\zeta} \frac{1 + z^+_{2k+1}\zeta}{1 + z^+_{2k}\zeta} \frac{1 + \overline{z^{++}_{2k+1}}\zeta}{1 + \overline{z^{++}_{2k}}\zeta} \frac{1 + z^{++}_{2k}\zeta}{1 + z^{++}_{2k+1}\zeta}.$$

According to the principle the Neumann function is determined by a product having simple zeros at the same trace points in $\mathbb{C}$ of the point $z \in D$. But in order to attain a convergent infinite product convergence generating factors have to be inserted. This motivates to introduce the function

$$Q(z,\zeta) = \prod_{k\in\mathbb{N}_0} \frac{\zeta - z^+_k}{(\zeta - i)} \frac{\zeta - \overline{z^+_k}}{(\zeta + i)} \frac{1 + z^+_k\zeta}{(1 + i\zeta)} \frac{1 + \overline{z^+_k}\zeta}{(1 - i\zeta)}$$
$$\times \frac{\zeta - z^{++}_k}{(\zeta - i)} \frac{\zeta - \overline{z^{++}_k}}{(\zeta + i)} \frac{1 + z^{++}_k\zeta}{(1 + i\zeta)} \frac{1 + \overline{z^{++}_k}\zeta}{(1 - i\zeta)}.$$

In order to show $N_1(z,\zeta) = -\log |Q(z,\zeta)|^2$ to serve as a Neumann function for the domain D the first three properties of the Neumann function will be verified.

However, the normalization condition is not expected to hold. Moreover, also the behavior of its normal derivative has to be studied when the other variable is located at the respective boundary part.

2. A Neumann function for the circular rectangle

Having once shown the uniform convergence of the product Q its order may be changed arbitrarily without changing its value. This means that any of the z_k^+-points can be chosen as starting point for the other points of the sequences, where just the orders of the sequences will be altered. This follows from the construction of the parqueting. This also is true if one of the $z_k^{++}-$points is taken as a starting point or even one of their reflections at the real or imaginary axes. Hence, if N_1 is a Neumann function for D the same expression is a Neumann function for any of the domains from the parqueting.

Lemma 2.1. *The product $Q(z,\zeta)$ is absolutely and uniformly convergent for $z,\zeta \in D$, representing an analytic function there with a simple pole at $\zeta = z$.*

The denominators from Q are introduced for convergence reasons. Their zeros $\pm i$ are lying outside of any of the domains. The domains are accumulating at these two points.

Proof. The infinite product with regard to the first factor is convergent as

$$\sum_{k\in\mathbb{N}_0}\Big[\frac{\overline{[\zeta - z_{2k}^+]}[\zeta - z_{2k+1}^+]}{|\zeta - i|^2} - 1\Big]$$

is. This is seen from

$$|i - z_k^+| \le |z_k^+ - m_k i| + m_k - 1, \quad |z_k^+\overline{z_{k+1}^+} - 1| \le |z_k^+ - i||z_{k+1}^+ - i| + |z_k^+ - i| + |z_{k+1}^+ - i|$$

and the estimates

$$0 < m_{2k+1} - 1 \le q_1^{2k}(m_1 - 1), 0 < m_{2k} - 1 \le q_1^{2(k-1)}(m_2 - 1), k \in \mathbb{N},$$

where

$$q_1 = \frac{m_1 + 1}{m_1 - 1}\frac{m_2 - 1}{m_2 + 1} < 1,$$

see [14], Lemma 2.2. The remaining products are similarly treatable. □

For calculating the normal derivative of $N_1(z,\zeta)$ with respect to ζ on ∂D up to the four corner points for any $z \in D$ the product is written as

$$Q = \prod_{k\in\mathbb{N}_0} Q_k,$$

$$Q_k = \frac{\zeta - z_k^+}{(\zeta - i)}\frac{\zeta - \overline{z_k^+}}{(\zeta + i)}\frac{1 + z_k^+\zeta}{(1 + i\zeta)}\frac{1 + \overline{z_k^+}\zeta}{(1 - i\zeta)}\frac{\zeta - z_k^{++}}{(\zeta - i)}\frac{\zeta - \overline{z_k^{++}}}{(\zeta + i)}\frac{1 + z_k^{++}\zeta}{(1 + i\zeta)}\frac{1 + \overline{z_k^{++}}\zeta}{(1 - i\zeta)},$$

where

$$\begin{aligned}\partial_\zeta \log Q_k =& \frac{1}{\zeta - z_k^+} + \frac{1}{\zeta - \overline{z_k^+}} + \frac{z_k^+}{1 + z_k^+\zeta} + \frac{\overline{z_k^+}}{1 + \overline{z_k^+}\zeta} \\ &+ \frac{1}{\zeta - z_k^{++}} + \frac{1}{\zeta - \overline{z_k^{++}}} + \frac{z_k^{++}}{1 + z_k^{++}\zeta} + \frac{\overline{z_k^{++}}}{1 + \overline{z_k^{++}}\zeta} \\ &- 2\Big[\frac{1}{\zeta - i} + \frac{1}{\zeta + i} + \frac{i}{1 + i\zeta} - \frac{i}{1 - i\zeta}\Big].\end{aligned}$$

Lemma 2.2. *On ∂D outside the corner points*

$$\partial_{\nu_\zeta} N_1(z,\zeta) = \begin{cases} 0, & \sqrt{2} < |\zeta - i\sqrt{3}|, \\ -4\sqrt{2}\frac{\sqrt{3}-1}{|\zeta - i|^2}, & |\zeta - i\sqrt{3}| = \sqrt{2}. \end{cases}$$

Proof. 1. On the segment of the real axis, where $\zeta - \overline{\zeta} = 0$ and $\partial_{\nu_\zeta} = -i(\partial_\zeta - \partial_{\overline{\zeta}})$ obviously $\mathrm{Im}\partial_\zeta \log Q_k = 0$ for any $k \in \mathbb{N}_0$ so that $\partial_{\nu_\zeta} N_1(z,\zeta) = -2\mathrm{Im}\partial_\zeta \log Q = 0$.

2. For $|\zeta - 1| = \sqrt{2}$, i.e. $\zeta(\overline{\zeta} - 1) = \overline{\zeta} + 1$, the expression $\partial_{\nu_\zeta} = \frac{\zeta - 1}{\sqrt{2}}\partial_\zeta + \frac{\overline{\zeta} - 1}{\sqrt{2}}\partial_{\overline{\zeta}}$ indicates $\partial_{\nu_\zeta} N_1(z,\zeta) = -2\mathrm{Re}\frac{\zeta - 1}{\sqrt{2}}\partial_\zeta \log Q = 0$, if $\mathrm{Re}(\zeta - 1)\partial_\zeta \log Q_k = 0$ for any $k \in \mathbb{N}_0$. This last equation follows from

$$\begin{aligned}&\frac{\zeta - 1}{\zeta - z_k^{++}} = 1 - \frac{\overline{z_k^+}(\overline{\zeta} - 1)}{1 + \overline{z_k^+}\zeta}, \quad \frac{\zeta - 1}{\zeta - \overline{z_k^{++}}} = 1 - \frac{z_k^+(\overline{\zeta} - 1)}{1 + z_k^+\overline{\zeta}}, \\ &\frac{z_k^{++}(\zeta - 1)}{1 + z_k^{++}\zeta} = 1 - \frac{\overline{\zeta} - 1}{\overline{\zeta} - \overline{z_k^+}}, \quad \frac{\overline{z_k^{++}}(\zeta - 1)}{1 + \overline{z_k^{++}}\zeta} = 1 - \frac{\overline{\zeta} - 1}{\overline{\zeta} - z_k^+}, \\ &\frac{\pm i}{1 \pm i\zeta} = \frac{1}{\zeta \mp i}, \quad \frac{\zeta - 1}{\zeta \mp i} = 1 - \frac{\overline{\zeta} - 1}{\overline{\zeta} \pm i}.\end{aligned}$$

3. For $|\zeta + 1| = \sqrt{2}$, i.e. $\zeta(\overline{\zeta} + 1) = 1 - \overline{\zeta}$ from $\partial_{\nu_\zeta} = \frac{\zeta + 1}{\sqrt{2}}\partial_\zeta + \frac{\overline{\zeta} + 1}{\sqrt{2}}\partial_{\overline{\zeta}}$ is seen $\partial_{\nu_\zeta} N_1(z,\zeta) = -2\mathrm{Re}\frac{\zeta + 1}{\sqrt{2}}\partial_\zeta \log Q = 0$, if $\mathrm{Re}(\zeta + 1)\partial_\zeta \log Q_k = 0$ for any $k \in \mathbb{N}_0$. This last equation holds because

$$\begin{aligned}&\frac{\zeta + 1}{\zeta - z_k^{++}} = 1 - \frac{\overline{\zeta} + 1}{\overline{\zeta} - \overline{z_k^+}}, \quad \frac{\zeta + 1}{\zeta - \overline{z_k^{++}}} = 1 - \frac{\overline{\zeta} + 1}{\overline{\zeta} - z_k^+}, \\ &\frac{z_k^{++}(\zeta + 1)}{1 + z_k^{++}\zeta} = 1 - \frac{\overline{z_k^+}(\overline{\zeta} + 1)}{1 + \overline{z_k^+}\zeta}, \quad \frac{\overline{z_k^{++}}(\zeta + 1)}{1 + \overline{z_k^{++}}\zeta} = 1 - \frac{z_k^+(\overline{\zeta} + 1)}{1 + z_k^+\overline{\zeta}}, \\ &\frac{\pm i}{1 \pm i\zeta} = \frac{1}{\zeta \mp i}, \quad \frac{\zeta + 1}{\zeta \mp i} = 1 - \frac{\overline{\zeta} + 1}{\overline{\zeta} \pm i}.\end{aligned}$$

4. For $|\zeta - i\sqrt{3}| = \sqrt{2}$, where $\zeta(\overline{\zeta} + im_1) = im_1\overline{\zeta} - 1$, from $\partial_{\nu_\zeta} = -\frac{\zeta - im_1}{\sqrt{2}}\partial_\zeta - \frac{\overline{\zeta}+im_1}{\sqrt{2}}\partial_{\overline{\zeta}}$ follows

$$\sqrt{2}\partial_{\nu_\zeta} N_1(z,\zeta) = \sum_{k\in\mathbb{N}_0} 2\mathrm{Re}\{(\zeta - im_1)\partial_\zeta \log|\widetilde{Q}(z_k^+,\zeta)\widetilde{Q}(z_k^{++},\zeta)|^2\}$$

with

$$\widetilde{Q}(z,\zeta) = \frac{\zeta - z}{\zeta - i}\frac{\zeta - \overline{z}}{\zeta + i}\frac{1 + z\zeta}{1 + i\zeta}\frac{1 + \overline{z}\zeta}{1 - i\zeta}. \tag{2.1}$$

From the relation $\zeta(\overline{\zeta} + im_1) = im_1\overline{\zeta} - 1$, follow

$$\frac{\zeta - im_1}{\zeta - z} = 1 - \frac{\overline{\zeta} + im_1}{\overline{\zeta} - \frac{1+im_1 z}{im_1 - z}}, \frac{z(\zeta - im_1)}{1 + z\zeta} = 1 - \frac{\overline{\zeta} + im_1}{\overline{\zeta} - \frac{z - im_1}{im_1 z + 1}}. \tag{2.2}$$

Observing

$$z_{k+1}^+ = \frac{im_{k+1}\overline{z_k^+} - 1}{\overline{z_k^+} + im_{k+1}},\ k\in\mathbb{N}_0,\ \ m_{k+1} = \frac{1 + m_1 m_k}{m_1 + m_k},\ k\in\mathbb{N},$$

the first relation in (2) implies for $z \in D$

$$\mathrm{Re}\Big[\frac{\zeta - im_1}{\zeta - z} + \frac{\zeta - im_1}{\zeta - z_1^+}\Big] = 1, \mathrm{Re}\Big[\frac{\zeta - im_1}{\zeta - \overline{z_k^+}} + \frac{\zeta - im_1}{\zeta - z_{k+2}^+}\Big] = 1,$$
$$\mathrm{Re}\Big[\frac{\zeta - im_1}{\zeta - z_0^{++}} + \frac{\zeta - im_1}{\zeta - z_1^{++}}\Big] = 1, \mathrm{Re}\Big[\frac{\zeta - im_1}{\zeta - \overline{z_k^{++}}} + \frac{\zeta - im_1}{\zeta - z_{k+2}^{++}}\Big] = 1,$$

and the second

$$\mathrm{Re}\Big[\frac{z(\zeta - im_1)}{1 + z\zeta} + \frac{z_1^+(\zeta - im_1)}{1 + z_1^+\zeta}\Big] = 1,\ \mathrm{Re}\Big[\frac{\overline{z_k^+}(\zeta - im_1)}{1 + \overline{z_k^+}\zeta} + \frac{z_{k+2}^+(\zeta - im_1)}{1 + z_{k+2}^+\zeta}\Big] = 1,$$
$$\mathrm{Re}\Big[\frac{z_0^{++}(\zeta - im_1)}{1 + z_0^{++}\zeta} + \frac{z_1^{++}(\zeta - im_1)}{1 + z_1^+\zeta}\Big] = 1,$$
$$\mathrm{Re}\Big[\frac{\overline{z_k^{++}}(\zeta - im_1)}{1 + \overline{z_k^{++}}\zeta} + \frac{z_{k+2}^{++}(\zeta - im_1)}{1 + z_{k+2}^{++}\zeta}\Big] = 1.$$

In the same way

$$\mathrm{Re}\Big[\frac{\zeta - im_1}{\zeta - i} + \frac{\zeta - im_1}{\zeta + i}\Big] = 1$$

is seen. Thus

$$\sqrt{2}\partial_{\nu_\zeta} N_1(z,\zeta) = \sum_{k\in\mathbb{N}_0} 2\mathrm{Re}\big[(\zeta - im_1)\partial_\zeta[\log\widetilde{Q}(z_k^+,\zeta) + \log\widetilde{Q}(z_k^{++},\zeta)]\big],$$

where, dropping the "+" and "++" superscripts,

$$\sum_{k\in\mathbb{N}_0} \mathrm{Re}\big[(\zeta - im_1)\partial_\zeta[\log \widetilde{Q}(z_k,\zeta)]\big]$$
$$= \sum_{k\in\mathbb{N}_0} \mathrm{Re}\big[(\zeta - im_1)\Big[\frac{1}{\zeta - z_k} + \frac{1}{\zeta - \overline{z_k}} + \frac{z_k}{1 + z_k\zeta} + \frac{\overline{z_k}}{1+\overline{z_k}\zeta} - \frac{2}{\zeta - i} - \frac{2}{\zeta + i}\Big]\big]$$
$$= \mathrm{Re}\big[(\zeta - im_1)\Big[\frac{1}{\zeta - z_0} + \frac{1}{\zeta - z_1} + \frac{z_0}{1 + z_0\zeta} + \frac{z_1}{1+z_1\zeta} - \frac{4}{\zeta - i}$$
$$+ \sum_{k\in\mathbb{N}_0}\Big\{\frac{1}{\zeta - z_{k+2}} + \frac{z_{k+2}}{1+z_{k+2}\zeta} - \frac{2}{\zeta - i} + \frac{1}{\zeta - \overline{z_k}} + \frac{\overline{z_k}}{1+\overline{z_k}\zeta} - \frac{2}{\zeta + i}\Big\}\Big]\big]$$
$$= 2 - 4\mathrm{Re}\frac{\zeta - im_1}{\zeta - i} = -4\frac{m_1 - 1}{|\zeta - i|^2}.$$

Hence, for $|\zeta - i\sqrt{3}| = \sqrt{2}$

$$\partial_{\nu_\zeta} N_1(z,\zeta) = -4\sqrt{2}\frac{\sqrt{3}-1}{|\zeta - i|^2}.$$

□

Remark 2.3. This lemma confirms that N_1 is a Neumann function for D: it is harmonic as a function of ζ up to the singular point $\zeta = z$, and $N_1(z,\zeta)+\log|\zeta - z|^2$ is harmonic in D, and its normal derivative is piecewise continuous in ζ on the smooth parts of ∂D and independent of $z \in D$.

Obviously, there are many Neumann functions of this kind, because of the freedom in introducing convergence ensuring factors. They are not required for finitely many factors of the infinite product. For any such choice another "Neumann" function will arise varying in their normal derivatives on the boundary. E.g. replacing $\widetilde{Q}(z,\zeta)$ from (2.1) by

$$\widetilde{\widetilde{Q}}(z,\zeta) = (\zeta - z)\frac{\zeta - \overline{z}}{\zeta + i}(1 + z\zeta)\frac{1 + \overline{z}\zeta}{1 - i\zeta}$$

for $z = z_k^+$, and $z = z_k^{++}$ just for $k = 0, 1$ leads to a Neumann function $\widetilde{N}_1(z,\zeta)$ for D. Its normal derivative turns out to be

$$\partial_{\nu_\zeta}\widetilde{N}_1(z,\zeta) = \begin{cases} -\frac{8}{|\zeta - i|^2}, \text{ for } \zeta - \overline{\zeta} = 0, \\ -2\sqrt{2}, \text{ for } |\zeta + 1| = \sqrt{2} \text{ and for } |\zeta - 1| = \sqrt{2}, \\ 2\sqrt{2}, \text{ for } |\zeta - i\sqrt{3}| = \sqrt{2}. \end{cases}$$

As $w \in C^2(D;\mathbb{C})\cap C^1(\overline{D};\mathbb{C})$ for any regular domains D is representable via the Neumann function as

$$w(z) = -\frac{1}{4\pi}\int_{\partial D}\{w(\zeta)\partial_{\nu_\zeta}N_1(z,\zeta) - \partial_{\nu_\zeta}w(\zeta)N_1(z,\zeta)\}ds_\zeta$$
$$-\frac{1}{\pi}\int_D w_{\zeta\overline{\zeta}}(\zeta)N_1(z,\zeta)d\xi d\eta,$$

for the circular rectangle D this representation is

$$w(z) = \frac{4}{\pi i}\int_{\{|\zeta - i\sqrt{3}|^2=2\}\cap\partial D} \frac{\sqrt{3}-1}{|\zeta - i|^2} w(\zeta)\frac{d\zeta}{\zeta - i\sqrt{3}} + \frac{1}{4\pi}\int_{\partial D} \partial_{\nu_\zeta} w(\zeta) N_1(z,\zeta) ds_\zeta$$
$$- \frac{1}{\pi}\int_D \partial_\zeta \partial_{\overline{\zeta}} w(\zeta) N_1(z,\zeta) d\xi d\eta.$$

2.1. Normal derivative with respect to the $z-$variable of the Neumann function. In order to see the connection of the Neumann representation formula with the solutions to the Neumann boundary value problem for the Poisson equation the normal derivatives of the Neumann function with respect to the variable z have to be investigated. According to the different parts of the boundary four cases are to be distinguished.

The Neumann function is decomposed as

$$N_1(z,\zeta) = N_1^+(z,\zeta) + N_1^{++}(z,\zeta),$$

where the terms related to the z_k^+ are separated from the ones with the z_k^{++}. In particular,

$$N_1^+(z,\zeta) = -\sum_{k\in\mathbb{N}_0} \log|Q(z_k^+,\zeta)|^2$$
$$= -\log|Q(z_0^+,\zeta)|^2 - \sum_{k\in\mathbb{N}} \log|Q(z_{2k-1}^+,\zeta)Q(z_{2k}^+,\zeta)|^2$$

and analogously for the second part replacing $^+$ with $^{++}$. Here

$$Q(z,\zeta) = \frac{\zeta - z}{\zeta - i}\frac{\overline{\zeta} - z}{\overline{\zeta} - i}\frac{1+z\zeta}{1+i\zeta}\frac{1+z\overline{\zeta}}{1+i\overline{\zeta}}.$$

Because of $\partial_z z_k^+ = 0$ for k odd and $\partial_z z_k^{++} = 0$ for k even

$$-\partial_z N_1^+(z,\zeta) = \partial_z \log Q(z_0^+,\zeta) + \sum_{k\in\mathbb{N}}\Big[\overline{\partial_{\overline{z}} \log Q(z_{2k-1}^+,\zeta)} + \partial_z \log Q(z_{2k}^+,\zeta)\Big]$$

and

$$-\partial_z N_1^{++}(z,\zeta) = \overline{\partial_{\overline{z}} \log Q(z_0^{++},\zeta)} + \sum_{k\in\mathbb{N}}\Big[\partial_z \log Q(z_{2k-1}^{++},\zeta) + \overline{\partial_{\overline{z}} \log Q(z_{2k}^{++},\zeta)}\Big].$$

2.1.1. The left-hand boundary part $\{|z-1|^2 = 2\}$.

2.1.1.i $N_1^+(z,\zeta)$.

For $z \in \mathbb{D}$

$$-2\text{Re}(z-1)\partial_z N_1^+(z,\zeta) = 2\text{Re}(z-1)\Big[\sum_{k\in\mathbb{N}_0} \partial_z \log|Q(z_k^+,\zeta)|^2\Big],$$

$$2\text{Re}(z-1)\partial_z \log|Q(z_0^+,\zeta)|^2 = -\Big[\frac{\zeta-1}{\zeta - z} + \overline{\frac{\zeta - 1}{\zeta - z}} - 1\Big] - \frac{|\zeta-1|^2 - |z-1|^2}{|\overline{\zeta} - z|^2}$$
$$+ \frac{|\zeta|^2(|z-1|^2-2) + (|\zeta-1|^2-2)}{|1+z\zeta|^2} + \frac{|\zeta|^2(|z-1|^2-2) + (|\zeta-1|^2-2)}{|1+z\overline{\zeta}|^2} + 4,$$

$$\partial_z \log |Q(z^+_{2k-1},\zeta)|^2 = \Big[-\frac{1}{\zeta - \overline{z^+_{2k-1}}} - \frac{1}{\overline{\zeta - z^+_{2k-1}}} + \frac{\zeta}{1+\overline{z^+_{2k-1}}\zeta} + \frac{\overline{\zeta}}{1+\overline{z^+_{2k-1}\zeta}}\Big]\partial_z \overline{z^+_{2k-1}}, \tag{2.3}$$

$$\partial_z \log |Q(z^+_{2k},\zeta)|^2 = \Big[-\frac{1}{\zeta - z^+_{2k}} - \frac{1}{\overline{\zeta} - z^+_{2k}} + \frac{\zeta}{1+z^+_{2k}\zeta} + \frac{\overline{\zeta}}{1+z^+_{2k}\overline{\zeta}}\Big]\partial_z z^+_{2k}. \tag{2.4}$$

Differentiating

$$\overline{z^+_{2k+1}} = -\frac{im_1\overline{z^+_{2k-1}}+1}{\overline{z^+_{2k-1}} - im_1}, \quad \overline{z^+_1} = -\frac{im_1 z+1}{z-im_1},$$

shows

$$\partial_z \overline{z^+_{2k+1}} = \overline{\partial_{\overline{z}} z^+_{2k+1}} = \frac{-2}{(z-im_1)^2}\prod_{\kappa=0}^{k-1}\frac{-2}{(\overline{z^+_{2\kappa+1}}-im_1)^2}.$$

Lemma 2.4. *For* $k \in \mathbb{N}$

$$\prod_{\kappa=0}^{k-1}(\overline{z^+_{2\kappa+1}} - im_1) = a_{k-1}\overline{z^+_1} + b_{k-1},$$

and

$$(z-im_1)\prod_{\kappa=0}^{k-1}(\overline{z^+_{2\kappa+1}} - im_1) = a_k z + b_k,$$

with

$$a_0 = 1, b_0 = -im_1, a_k = -im_1 a_{k-1} + b_{k-1}, b_k = -a_{k-1} - im_1 b_{k-1}.$$

An induction argument ensures these formulas.

The first summand in (2.3)

$$2\mathrm{Re}\frac{z-1}{\zeta - \overline{z^+_{2k-1}}}\partial_z\overline{z^+_{2k-1}} = 2\mathrm{Re}\frac{(-2)^k(z-1)}{(z-im_1)\prod_{\kappa=0}^{k-2}(\overline{z^+_{2\kappa+1}}-im_1)}$$
$$\times\frac{1}{\big[\zeta(\overline{z^+_{2k-1}}-im_1)+(im_1\overline{z^+_{2k-1}}+1)\big](z-im_1)\prod_{\kappa=0}^{k-3}(\overline{z^+_{2\kappa+1}}-im_1)}$$
$$= 2\mathrm{Re}\frac{(-2)^k(z-1)}{(a_{k-1}z+b_{k-1})[(a_{k-1}z+b_{k-1})\zeta - b_{k-1}z + a_{k-1}]}$$

by partial fraction turns out to be

$$2\mathrm{Re}\Big[\frac{(a_{k-1}+b_{k-1})\zeta + a_{k-1} - b_{k-1}}{[(a_{k-1}\zeta - b_{k-1})z + a_{k-1} + b_{k-1}\zeta]} - \frac{a_{k-1}+b_{k-1}}{(a_{k-1}z+b_{k-1})}\Big]. \tag{2.5}$$

Calculations of these terms are enabled by some particular properties of the coefficients involved.

Lemma 2.5. *The coefficients from Lemma 2.4 satisfy for* $k \in \mathbb{N}_0$

$$\overline{a_k} b_k + a_k \overline{b_k} = 0, i\overline{a_k} b_k = |a_k||b_k|, |b_k|^2 - |a_k|^2 = 2^{k+1}, a_k^2 + b_k^2 = (-2)^{k+1},$$
$$|a_k| + |b_k| = (1+m_1)^{k+1}, \ |b_k| - |a_k| = (m_1 - 1)^{k+1}.$$

Remark 2.6. The recursive relations for the coefficients a_k, b_k can be used to determine the predecessors from the followers via

$$2a_{k-1} = im_1 a_k + b_k, \ \ 2b_{k-1} = im_1 b_k - a_k; \ \ a_{-1} = 0, \ \ b_{-1} = 1.$$

Obviously,

$$|a_k| < |b_k|, 2^{k+1} \le |b_k|$$

hold.

Proof. It is convenient to rewrite the recursion relations for the coefficients as

$$\widehat{a}_{2k} = 3\widehat{a}_{2k-1} + \widehat{b}_{2k-1}, \widehat{a}_{2k+1} = \widehat{a}_{2k} + \widehat{b}_{2k}, \widehat{b}_{2k} = \widehat{a}_{2k-1} + \widehat{b}_{2k-1}, \widehat{b}_{2k+1} = \widehat{a}_{2k} + 3\widehat{b}_{2k},$$

where

$$a_{2k} = (-1)^k \widehat{a}_{2k}, \ a_{2k+1} = (-1)^{k+1} im_1 \widehat{a}_{2k+1},$$
$$b_{2k} = (-1)^{k+1} im_1 \widehat{b}_{2k}, \ b_{2k+1} = (-1)^{k+1} \widehat{b}_{2k+1},$$

with real positive $\widehat{a}'s, \widehat{b}'s$, in particular $\widehat{a}_0 = \widehat{b}_0 = 1, \widehat{a}_1 = 2, \widehat{b}_1 = 4$. As

$$|b_{2k}| = m_1 \widehat{b}_{2k} = m_1[2\widehat{a}_{2(k-1)} + 4\widehat{b}_{2(k-1)}] > 4m_1 \widehat{b}_{2(k-1)} = 4|b_{2(k-1)}| > 4^k |\widehat{b}_0| = 4^k$$

similarly also $|b_{2k+1}| > 4^k$ is seen. Moreover, as well

$$\begin{aligned} |b_k|^2 - |a_k|^2 &= |a_{k-1} + im_1 b_{k-1}|^2 - |b_{k-1} - im_1 a_{k-1}|^2 \\ &= 2[|b_{k-1}|^2 - |a_{k-1}|^2] = 2^k[|b_0|^2 - |a_0|^2] = 2^{k+1} \end{aligned}$$

as

$$|a_{k-1}| \le |a_k|, \ \ |b_{k-1}| \le |b_k|$$

hold. The manipulation

$$\begin{aligned} |a_{2k}| + |b_{2k}| = \widehat{a}_{2k} + m_1 \widehat{b}_{2k} &= (4 + 2m_1)(\widehat{a}_{2k-2} + m_1 \widehat{b}_{2k-2}) \\ &= (4 + 2m_1)(|a_{2k-2}| + |b_{2k-2}|) \end{aligned}$$

and the analogous relation for odd indices $2k+1$ resulting in

$$|a_{2k}| + |b_{2k}| = (4 + 2m_1)^k (1 + m_1) = (1 + m_1)^{2k+1},$$
$$|a_{2k+1}| + |b_{2k+1}| = (4 + 2m_1)^{k+1} = (1 + m_1)^{2k+2},$$

i.e. $|a_k| + |b_k| = (1 + m_1)^{k+1}$, provide bounds for $|a_k| + |b_k|$. In the same way

$$|b_{2k}| - |a_{2k}| = (4 - 2m_1)^k (m_1 - 1) = (m_1 - 1)^{2k+1},$$
$$|b_{2k+1}| - |a_{2k+1}| = (4 - 2m_1)^{k+1} = (m_1 - 1)^{2k+2}$$

are seen. The $|a_k|$ and $|b_k|$ and hence the coefficients a_k, b_k themselves can explicitly be determined from the respective two sets of equations as

$$2a_{2k} = (-1)^{k+1}[(m_1+1)^{2k+1} - (m_1-1)^{2k+1}], \tag{2.6}$$
$$2b_{2k} = (-1)^k i[(m_1+1)^{2k+1} + (m_1-1)^{2k+1}],$$
$$2a_{2k+1} = (-1)^k i[(m_1+1)^{2(k+1)} - (m_1-1)^{2(k+1)}], \tag{2.7}$$
$$2b_{2k+1} = (-1)^k[(m_1+1)^{2(k+1)} + (m_1-1)^{2(k+1)}].$$

Also

$$m_1|a_k| - |b_k| = 2|a_{k-1}|, \quad m_1|b_k| - |a_k| = 2|b_{k-1}|$$

are available. □

For estimating $|a_k(1+z\zeta) + b_k(\zeta - z)|$ from below for $z, \zeta \in \overline{D}$ the inequalities

$$0 \le -i(z-\overline{z}) \le -im_1(z-\overline{z}) \le 1 + |z|^2,$$

are used and also

$$|z| \le |z_1| \text{ with } z_1 = \frac{1}{2}(m_1-1)(1+i), \quad |z - im_1| \le 2(3-\sqrt{2})$$

for $z \in D$, where the latter two follow from the facts that z_1 (as also $-\overline{z_1}$) is an extremal point for $|z|$ in the domain D, star-shaped with respect to the origin, and $\pm(\sqrt{2}-1)$ are extremal for $|z - im_1|$.

Corollary 2.7. *For any z and $k \in \mathbb{N}_0$ the relation*

$$|(a_k+b_k)z+b_k-a_k|^2 - 2|a_kz+b_k|^2 = 2^{k+1}(|z+1|^2-2) = 2^{k+1}[2-|z-1|^2+2(|z|^2-1)]$$

holds.

This is a consequence from Lemma 2.5.

Lemma 2.8. *For real numbers $\alpha_0, \beta_0, \kappa_0, \lambda_0$ and $n \in \mathbb{N}$ the relations*

$$\alpha_0\widehat{a}_{2n} + \beta_0\widehat{b}_{2n} = \alpha_{2n}\widehat{a}_0 + \beta_{2n}\widehat{b}_0 = \alpha_{2n} + \beta_{2n},$$

$$\kappa_0\widehat{a}_{2n+1} + \lambda_0\widehat{b}_{2n+1} = \kappa_{2n}\widehat{a}_{-1} + \lambda_{2n}\widehat{b}_{-1} = \lambda_{2n}$$

with

$$\alpha_{2\nu} = 2(2\alpha_{2\nu-2} + \beta_{2\nu-2}), \beta_{2\nu} = 2(3\alpha_{2\nu-2} + 2\beta_{2\nu-2}),$$
$$\kappa_{2\nu} = 2(2\kappa_{2\nu-2} + 3\lambda_{2\nu-2}), \lambda_{2\nu} = 2(\kappa_{2\nu-2} + 2\lambda_{2\nu-2})$$

for $0 \le \nu \le n$ hold. Moreover, for these ν

$$\alpha_{2\nu} = 2^\nu(\gamma_{2\nu}\alpha_0 + \delta_{2\nu}\beta_0), \beta_{2\nu} = 2^\nu(3\delta_{2\nu}\alpha_0 + \gamma_{2\nu}\beta_0),$$

$$\kappa_{2\nu} = 2^\nu(2\gamma_{2\nu}\kappa_0 + 3\delta_{2\nu}\lambda_0), \lambda_{2\nu} = 2^\nu(\delta_{2\nu}\kappa_0 + \gamma_{2\nu}\lambda_0),$$

where with $\gamma_2 = 2, \delta_2 = 1$ and for $1 < \nu \le n$

$$\gamma_{2\nu} = 2\gamma_{2\nu-2} + 3\delta_{2\nu-2}, \delta_{2\nu} = \gamma_{2\nu-2} + 2\delta_{2\nu-2}$$

are valid.

Proof. Exemplarily, the case of even indices is treated. Applying the recursion relations for the $\widehat{a}'s, \widehat{b}'s$ from the last proof

$$\alpha_0\widehat{a}_{2n} + \beta_0\widehat{b}_{2n} = 2(2\alpha_0 + \beta_0)\widehat{a}_{2n-2} + 2(3\alpha_0 + 2\beta_0)\widehat{b}_{2n-2} = \alpha_2\widehat{a}_{2n-2} + \beta_2\widehat{b}_{2n-2}$$

is deduced. Arguing inductively shows the first equality. The second follows because $\widehat{a}_0 = \widehat{b}_0 = 1$. In the odd index case arguing is similar where $\widehat{a}_{-1} = 0, \widehat{b}_{-1} = 1$ is to be observed.

Expressing the $\alpha's, \beta's$ by α_0, β_0 again induction is helpful, starting with

$$\alpha_2 = 2(2\alpha_0 + \beta_0), \beta_2 = 2(3\alpha_0 + 2\beta_0).$$

The resulting formulas for the $\gamma's, \delta's$ can be extended also to $\nu = 1$ when defining $\gamma_0 = 1, \delta_0 = 0$.

Starting point for the $\kappa's, \lambda's$ are likewise

$$\kappa_2 = 2(2\kappa_0 + 3\lambda_0), \lambda_2 = 2(\kappa_0 + 2\lambda_0).$$

□

The $(\gamma_{2\nu}), (\delta_{2\nu})$ are strongly increasing sequences of natural numbers. They are listed in [15] as $A001075$ and $A001353$. Some related sequences are studied in [16, 17].

Corollary 2.9. *For* $\nu \in \mathbb{N}$

$$3^\nu < \gamma_{2n}, 3^{\nu-1} < \delta_{2\nu}.$$

Proof. Setting $\gamma_{-2} = -2, \delta_{-2} = -1$ from the defining equations of the sequences inductively

$$\gamma_{2\nu} = 4\gamma_{2\nu-2} - \gamma_{2\nu-4}, \delta_{2\nu} = 4\delta_{2\nu-2} - \delta_{2\nu-4}$$

is proved. From here the monotony is seen and the estimates

$$\gamma_{2\nu} > 3\gamma_{2\nu-2} > 3^\nu\gamma_0, \delta_{2\nu} > 3\delta_{2\nu-2} > 3^{\nu-1}\delta_2 = 3^{\nu-1}.$$

□

After these investigations of the coefficients the two terms in (2.5) are further handled. They are rewritten as

$$2\mathrm{Re}\frac{a_{k-1} + b_{k-1}}{(a_{k-1}z + b_{k-1})} = \Big[\frac{a_{k-1} + b_{k-1}}{a_{k-1}z + b_{k-1}} + \frac{\overline{a_{k-1}}(1-\overline{z})}{\overline{a_{k-1}z + b_{k-1}}} + 1\Big],$$

$$\begin{aligned} &2\mathrm{Re}\frac{(a_{k-1} + b_{k-1})\zeta + a_{k-1} - b_{k-1}}{[(a_{k-1}z + b_{k-1})\zeta + a_{k-1} - b_{k-1}z]} \\ &= \Big[\frac{(a_{k-1} + b_{k-1})\zeta + a_{k-1} - b_{k-1}}{(a_{k-1}z + b_{k-1})\zeta + a_{k-1} - b_{k-1}z} + \frac{(\overline{a_{k-1}\zeta - b_{k-1}})(1-\overline{z})}{\overline{(a_{k-1}z + b_{k-1})\zeta + a_{k-1} - b_{k-1}z}} + 1\Big] \end{aligned}$$

and subtracting the first from the latter gives

$$2\mathrm{Re}\Big[\frac{z-1}{\overline{\zeta - z^+_{2k-1}}}\partial_z \overline{z^+_{2k-1}}\Big]$$
$$= \Big[\frac{|(a_{k-1}+b_{k-1})\zeta + a_{k-1} - b_{k-1}|^2 - |a_{k-1}\zeta - b_{k-1}|^2|z-1|^2}{|a_{k-1}(z\zeta+1)+b_{k-1}(\zeta - z)|^2}$$
$$- \frac{|a_{k-1}+b_{k-1}|^2 - |a_{k-1}|^2|z-1|^2}{|a_{k-1}z+b_{k-1}|^2}\Big].$$

In the same way

$$2\mathrm{Re}\Big[\frac{z-1}{\overline{\overline{\zeta} - z^+_{2k-1}}}\partial_z \overline{z^+_{2k-1}}\Big]$$
$$= \Big[\frac{|(a_{k-1}+b_{k-1})\overline{\zeta} + a_{k-1} - b_{k-1}|^2 - |a_{k-1}\overline{\zeta} - b_{k-1}|^2|z-1|^2}{|(a_{k-1}(z\overline{\zeta}+1)) + b_{k-1}(\overline{\zeta} - z)|^2}$$
$$- \frac{|a_{k-1}+b_{k-1}|^2 - |a_{k-1}|^2|z-1|^2}{|a_{k-1}z+b_{k-1}|^2}\Big]$$

is deduced.

For simplifying

$$|(a_{k-1}+b_{k-1})\zeta + a_{k-1} - b_{k-1}|^2 - 2|a_{k-1}\zeta - b_{k-1}|^2 = 2^k(|\zeta - 1|^2 - 2),$$

a consequence from Lemma 2.5 is used.

In an analogue way the remaining two terms from (2.3) can be treated leading to

$$-2\mathrm{Re}[(z-1)\partial_z \log|Q(z^+_{2k-1},\zeta)|^2$$
$$= \Big[\frac{|a_{k-1}\zeta - b_{k-1}|^2(2-|z-1|^2) - 2^k(2-|\zeta-1|^2)}{|(a_{k-1}z+b_{k-1})\zeta + a_{k-1} - b_{k-1}z|^2} - \frac{|a_{k-1}|^2(2-|z-1|^2)+2^k}{|a_{k-1}z+b_{k-1}|^2}\Big]$$
$$+ \Big[\frac{|a_{k-1}\overline{\zeta} - b_{k-1}|^2(2-|z-1|^2) - 2^k(2-|\zeta-1|^2)}{|(a_{k-1}z+b_{k-1})\overline{\zeta} + a_{k-1} - b_{k-1}z|^2} - \frac{|a_{k-1}|^2(2-|z-1|^2)+2^k}{|a_{k-1}z+b_{k-1}|^2}\Big]$$
$$+ \Big[\frac{|a_{k-1} + b_{k-1}\zeta|^2(2-|z-1|^2) + 2^k(2-|\zeta-1|^2)}{|a_{k-1}z+b_{k-1} + (b_{k-1}z - a_{k-1})\zeta|^2} - \frac{|a_{k-1}|^2(2-|z-1|^2)+2^k}{|a_{k-1}z+b_{k-1}|^2}\Big]$$
$$+ \Big[\frac{|a_{k-1} + b_{k-1}\overline{\zeta}|^2(2-|z-1|^2) + 2^k(2-|\zeta-1|^2)}{|a_{k-1}z+b_{k-1} + (b_{k-1}z - a_{k-1})\overline{\zeta}|^2} - \frac{|a_{k-1}|^2(2-|z-1|^2)+2^k}{|a_{k-1}z+b_{k-1}|^2}\Big].$$

For $|\zeta - 1|^2 = 2$ this is

$$-2\mathrm{Re}[(z-1)\partial_z \log|Q(z^+_{2k-1},\zeta)|^2] = -\frac{2^{k+2}}{|a_{k-1}z+b_{k-1}|^2} - \Big\{\frac{|a_{k-1}|^2}{|a_{k-1}z+b_{k-1}|^2}$$
$$- \Big[\frac{|a_{k-1}\zeta - b_{k-1}|^2}{|(a_{k-1}z+b_{k-1})\zeta + a_{k-1} - b_{k-1}z|^2} + \frac{|a_{k-1}\overline{\zeta} - b_{k-1}|^2}{|(a_{k-1}z+b_{k-1})\overline{\zeta} + a_{k-1} - b_{k-1}z|^2}$$
$$+ \frac{|a_{k-1}+b_{k-1}\zeta|^2}{|a_{k-1}z+b_{k-1}+(b_{k-1}z - a_{k-1})\zeta|^2} + \frac{|a_{k-1}+b_{k-1}\overline{\zeta}|^2}{|a_{k-1}z+b_{k-1}+(b_{k-1}z - a_{k-1})\overline{\zeta}||^2}\Big]\Big\}$$
$$\times (2 - |z-1|^2).$$

For summing these terms up for $k \in \mathbb{N}$ estimation is needed for ensuring convergence.

Lemma 2.10. *For $z \in D$ and $k \in \mathbb{N}$*

$$6^k < 2^2 3^4 |a_{k-1} z + b_{k-1}|^2,$$
$$6^k < 2^2 3^6 |a_{k-1}(1 + z\zeta) + b_{k-1}(\zeta - z)|^2.$$

Proof. i. From $|z| \leq |z_1| = \sqrt{2 - m_1} < \frac{1}{m_1}$ the estimation

$$|a_{k-1} z + b_{k-1}| > |b_{k-1}| - \frac{1}{m_1}|a_{k-1}|$$

is seen. For further estimation as well the notations from the proof of Lemma 2.5 as the results from Lemma 2.8 and Corollary 2.9 are used. In the case of an even index $k - 1$ denoting $\alpha_0 = -\frac{1}{m_1}, \beta_0 = m_1$, Lemma 2.5 and Corollary 2.9 supply

$$\begin{aligned}|b_{2n}| - |a_{2n}||z| &> m_1 \widehat{b}_{2n} - \frac{1}{m_1}\widehat{a}_{2n} = \alpha_0 \widehat{a}_{2n} + \beta_0 \widehat{b}_{2n} = \alpha_{2n}\widehat{a}_0 + \beta_{2n}\widehat{b}_0 = \alpha_{2n} + \beta_{2n} \\ &= 2^n(\gamma_{2n}\alpha_0 + \delta_{2n}\beta_0 + 3\delta_{2n}\alpha_0 + \gamma_{2n}\beta_0) \\ &= 2^n[\gamma_{2n}(m_1 - \frac{1}{m_1}) + \delta_{2n}(m_1 - m_1)] > 2^n\gamma_{2n} > 2^n 3^n.\end{aligned}$$

In the same way

$$\begin{aligned}|b_{2n+1}| - \frac{1}{m_1}|a_{2n+1}| &= \widehat{b}_{2n+1} - \widehat{a}_{2n+1} = \lambda_{2n} = 2^n(\gamma_{2n} - \delta_{2n}) \\ &= 2^n(\gamma_{2n-2} + \delta_{2n-2}) > 2^n(3^{n-1} + 3^{n-2}) > 2^n 3^{n-1}\end{aligned}$$

is deduced, so that in general

$$(|b_n| - \frac{1}{m_1}|a_n|)^2 > 2^n 3^n \frac{1}{54}$$

and hence

$$2^2 3^4 |a_{k-1} z + b_{k-1}|^2 > 6^n$$

follow.

ii. For $z, \zeta \in D$

$$|1 + z\zeta| \geq 1 - |z_1|^2 = m_1 - 1, \quad |\zeta - z|^2 \leq |z_1 - (\sqrt{2} - 1)|^2 = 4 - \sqrt{2} - \sqrt{6}.$$

The bounds, $\hat{\alpha}_0 = m_1 - 1$ and $\hat{\beta}_0 = \sqrt{4 - \sqrt{2} - \sqrt{6}}$, are related by $0 < \hat{\alpha}_0 - m_1\hat{\beta}_0$. As before even and odd indices have to be treated separately.

$$\begin{aligned}&|a_{2n}||1 + z\zeta| - |b_{2n}||\zeta - z| \geq \hat{\alpha}_0|a_{2n}| - \hat{\beta}_0|b_{2n}| = \hat{\alpha}_0\widehat{a}_{2n} - m_i\hat{\beta}_0\widehat{b}_{2n} \\ &= \hat{\alpha}_{2n} + \hat{\beta}_{2n} = 2^n[\hat{\alpha}_0(\gamma_{2n} + 3\delta_{2n}) - m_1\hat{\beta}_0(\gamma_{2n} + \delta_{2n})] \\ &= 2^n[(\hat{\alpha}_0 - m_1\hat{\beta}_0)\gamma_{2n} + (3\hat{\alpha}_0 - m_1\hat{\beta}_0)\delta_{2n}] > 2^n[2\hat{\alpha}_0\delta_{2n}] > 2^n\delta_{2n} > 2^n 3^{n-1}.\end{aligned}$$

With $\hat{\kappa}_0 = m_1\hat{\alpha}_0, \hat{\lambda}_0 = -\hat{\beta}_0$

$$\begin{aligned}&|a_{2n+1}||1 + z\zeta| - |b_{2n+1}||\zeta - z| \geq \hat{\alpha}_0|a_{2n+1}| - \hat{\beta}_0|b_{2n+1}| = \hat{\kappa}_0\widehat{a}_{2n+1} - \hat{\lambda}_0\widehat{b}_{2n+1} \\ &= \hat{\lambda}_{2n} = 2^n[\hat{\lambda}_0\gamma_{2n} + \hat{\kappa}_0\delta_{2n}] = 2^n[m_1\hat{\alpha}_0\delta_{2n} - \hat{\beta}_0\gamma_{2n}] \\ &= 2^n[(m_1\hat{\alpha}_0 - 2\hat{\beta}_0)\gamma_{2n-2} + (2m_1\hat{\alpha}_0 - 3\hat{\beta}_0)\delta_{2n-2}] > 2^n m_1\hat{\alpha}_0\delta_{2n-2} > 2^n 3^{n-2}.\end{aligned}$$

Thus,

$$2^2 3^6 |a_{k-1}(1+z\zeta)+b_{k-1}(\zeta-z)|^2 \geq 6^k.$$

□

These estimations guarantee the convergence of the sums when summing up the

$$2\mathrm{Re}[(z-1)\partial_z \log|Q(z^+_{2k-1},\zeta)|^2.$$

Next the terms in (2.4) are calculated on the basis of

$$z^+_0 = z, z^+_{2k+2} = \frac{im_1 z^+_{2k} - 1}{z^+_{2k} + im_1}, d_z z^+_{2k} = \prod_{\kappa=0}^{k-1} \frac{-2}{(z^+_{2\kappa}+im_1)^2}, k \in \mathbb{N}.$$

At first the term for $k=0$ is looked at,

$$2\mathrm{Re}\Big[\frac{z-1}{\zeta-z} + \frac{z-1}{\overline{\zeta}-z} - \frac{(z-1)\zeta}{1+z\zeta} - \frac{(z-1)\overline{\zeta}}{1+z\overline{\zeta}}\Big].$$

It turns out to be

$$\begin{aligned}&\Big[\frac{\zeta-1}{\zeta-z}+\frac{\overline{\zeta}-1}{\overline{\zeta-z}}-1\Big]+\frac{|\zeta-1|^2-|z-1|^2}{|\overline{\zeta}-z|^2}\\ \Sigma &- \frac{|\zeta|^2(|z-1|^2-2)+|\zeta-1|^2-2}{|1+z\zeta|^2} - \frac{|\zeta|^2(|z-1|^2-2)+|\zeta-1|^2-2}{|1+z\overline{\zeta}|^2} - 4,\end{aligned} \tag{2.8}$$

where the first term is up to the factor $\frac{1}{\sqrt{2}}$ the Poisson kernel for the disc $|z-1|^2 < 2$.

Lemma 2.11. *For* $k \in \mathbb{N}_0$

$$\prod_{\kappa=0}^{k} (z^+_{2\kappa}+im_1) = \widetilde{a}_k z + \widetilde{b}_k,$$

with

$$\widetilde{a}_0 = 1, \widetilde{b}_0 = im_1, \widetilde{a}_{k+1} = im_1\widetilde{a}_k + \widetilde{b}_k, \widetilde{b}_{k+1} = -\widetilde{a}_k + im_1\widetilde{b}_k,$$

satisfying

$$\widetilde{a}_k\overline{\widetilde{b}_k} + \overline{\widetilde{a}_k}\widetilde{b}_k = 0, \widetilde{a}_k^2 + \widetilde{b}_k^2 = (-2)^{k+1}, |\widetilde{b}_k|^2 - |\widetilde{a}_k|^2 = 2^{k+1}.$$

Moreover, $\widetilde{a}_k = \overline{a_k}, \widetilde{b}_k = \overline{b_k}$ *with the coefficients* a_k, b_k *from Lemma 4.*

As for Lemma 2.4 the proof is by induction argumentation.

Taking for $k \in \mathbb{N}$ the real part of

$$\begin{aligned}\frac{z-1}{\zeta - z^+_{2k}} d_z z^+_{2k} &= \frac{(z-1)(-2)^k}{(\widetilde{a}_{k-1}z+\widetilde{b}_{k-1})[(\widetilde{a}_{k-1}z+\widetilde{b}_{k-1})\zeta+\widetilde{a}_{k-1}-\widetilde{b}_{k-1}z]}\\ &= \frac{(\widetilde{a}_{k-1}+\widetilde{b}_{k-1})\zeta+\widetilde{a}_{k-1}-\widetilde{b}_{k-1}}{(\widetilde{a}_{k-1}z+\widetilde{b}_{k-1})\zeta+\widetilde{a}_{k-1}-\widetilde{b}_{k-1}z} - \frac{\widetilde{a}_{k-1}+\widetilde{b}_{k-1}}{\widetilde{a}_{k-1}z+\widetilde{b}_{k-1}}\end{aligned}$$

leads to

$$- 2\text{Re}\frac{z-1}{\zeta - z_{2k}^{+}} d_z z_{2k}^{+} = \frac{|\widetilde{a}_{k-1} + \widetilde{b}_{k-1}|^2 - |\widetilde{a}_{k-1}|^2|z-1|^2}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2}$$
$$- \frac{|(\widetilde{a}_{k-1} + \widetilde{b}_{k-1})\zeta + \widetilde{a}_{k-1} - \widetilde{b}_{k-1}|^2 - |\widetilde{a}_{k-1}\zeta - \widetilde{b}_{k-1}|^2|z-1|^2}{|\widetilde{a}_{k-1}(z\zeta + 1) + \widetilde{b}_{k-1}(\zeta - z)|^2}.$$

In the same way

$$2\text{Re}\frac{\zeta(z-1)}{1 + z_{2k}^{+}\zeta} d_z z_{2k}^{+} = \frac{|\widetilde{a}_{k-1}|^2(2 - |z-1|^2) + 2^k}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2}$$
$$- \frac{|\widetilde{a}_{k-1} + \widetilde{b}_{k-1}\zeta|^2(2 - |z-1|^2) + 2^k(2 - |\zeta - 1|^2)}{|(\widetilde{b}_{k-1}z - \widetilde{a}_{k-1})\zeta + \widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2}$$

is attained giving

$$- 2\text{Re}(z-1)\partial_z \log|Q(z_{2k}^{+}, \zeta)|^2$$
$$= \frac{|\widetilde{a}_{k-1}|^2(|z-1|^2 - 2) - 2^k}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2} + \frac{|\widetilde{a}_{k-1}\zeta - \widetilde{b}_{k-1}|^2(2 - |z-1|^2) - 2^k(2 - |\zeta - 1|^2)}{|(\widetilde{a}_{k-1}z + \widetilde{b}_{k-1})\zeta + \widetilde{a}_{k-1} - \widetilde{b}_{k-1}z|^2}$$
$$+ \frac{|\widetilde{a}_{k-1}|^2(|z-1|^2 - 2) - 2^k}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2} + \frac{|\widetilde{a}_{k-1}\overline{\zeta} - \widetilde{b}_{k-1}|^2(2 - |z-1|^2) - 2^k(2 - |\zeta - 1|^2)}{|(\widetilde{a}_{k-1}z + \widetilde{b}_{k-1})\overline{\zeta} + \widetilde{a}_{k-1} - \widetilde{b}_{k-1}z|^2}$$
$$+ \frac{|\widetilde{a}_{k-1}|^2(|z-1|^2 - 2) - 2^k}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2} + \frac{|\widetilde{a}_{k-1} + \widetilde{b}_{k-1}\zeta|^2(2 - |z-1|^2) + 2^k(2 - |\zeta - 1|^2)}{|(\widetilde{b}_{k-1}z - \widetilde{a}_{k-1})\zeta + \widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2}$$
$$+ \frac{|\widetilde{a}_{k-1}|^2(|z-1|^2 - 2) - 2^k}{|\widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2} + \frac{|\widetilde{a}_{k-1} + \widetilde{b}_{k-1}\overline{\zeta}|^2(2 - |z-1|^2) + 2^k(2 - |\zeta - 1|^2)}{|(\widetilde{b}_{k-1}z - \widetilde{a}_{k-1})\overline{\zeta} + \widetilde{a}_{k-1}z + \widetilde{b}_{k-1}|^2}.$$

2.1.1.ii $N_1^{++}(z, \zeta)$.
Next the last part from $N_1(z, \zeta)$ is handled. Because of the relations

$$z_k^{++} = \frac{1 - \overline{z_k^{+}}}{1 + \overline{z_k^{+}}}$$

the additional factors

$$\frac{-2}{(1 + \overline{z_{2k-1}^{+}})^2}, \quad \frac{-2}{(1 + z_{2k}^{+})^2},$$

respectively, have to be added, when calculating the $z-$derivatives of the terms of $N_1^{++}(z, \zeta)$ compared with the ones from $N_1^{+}(z, \zeta)$.

Applying Lemma 2.11

$$
\begin{aligned}
-(z-1)\partial_z \log(\zeta - z^{++}_{2k-1})
&= \frac{z-1}{\zeta - z^{++}_{2k-1}} \frac{(-2)^{k+1}}{(1+\overline{z^{+}_{2k-1}})^2 (z-im_1)^2 \prod_{\kappa=0}^{k-2}(\overline{z^{+}_{2\kappa+1}} - im_1)^2} \\
&= \frac{(-2)^{k+1}(z-1)}{\zeta[(a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}] - [(a_{k-1}-b_{k-1})z + a_{k-1} + b_{k-1}]} \\
&\times \frac{1}{(a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}} \\
&= \frac{2(b_{k-1}\zeta - a_{k-1})}{\zeta[(a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}] - [(a_{k-1}-b_{k-1})z + a_{k-1} + b_{k-1}]} \\
&- \frac{2b_{k-1}}{(a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}}
\end{aligned}
$$

is seen. Manipulations as for the terms in (2.5) give

$$
\begin{aligned}
&2\mathrm{Re}\frac{2b}{(a+b)z+b-a} = 1 - \frac{|a+b|^2|z-1|^2 - 4|b|^2}{|(a+b)z+b-a|^2}, \\
&2\mathrm{Re}\frac{2(b\zeta - a)}{\zeta[(a+b)z+b-a] - [(a-b)z+a+b]} \\
&\quad = 1 - \frac{|(a+b)\zeta + b - a|^2|z-1|^2 - 4|b\zeta - a|^2}{|\zeta[(a+b)z+b-a] - [(a-b)z+a+b]|^2}.
\end{aligned}
$$

Here again the indices $k-1$ were dropped. The relations

$$
|b_{k-1}|^2 - |a_{k-1}|^2 = 2^k, \quad a_{k-1}\overline{b_{k-1}} + \overline{a_{k-1}}b_{k-1} = 0,
$$

see Lemma 2.8, imply

$$
\begin{aligned}
&|a+b|^2|z-1|^2 - 4|b|^2 = |a+b|^2(|z-1|^2 - 2) - 2^{k+1}, \\
&|(a+b)\zeta + b - a|^2|z-1|^2 - 4|b\zeta - a|^2 = \\
&|(a+b)\zeta + b - a|^2(|z-1|^2 - 2) - 2^{k+1}(|\zeta - 1|^2 - 2), \\
&|(a-b)\zeta + a + b|^2|z-1|^2 - 4|a\zeta + b|^2 = \\
&|(a-b)\zeta + a + b|^2(|z-1|^2 - 2) + 2^{k+1}(|\zeta - 1|^2 - 2).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(z-1)\partial_z \log(1 + z^{++}_{2k-1}\zeta) &= \frac{2b_{k-1}}{(a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}} \\
&- \frac{2(a_{k-1}\zeta + b_{k-1})}{\zeta[(a_{k-1}-b_{k-1})z + a_{k-1} + b_{k-1}] + (a_{k-1}+b_{k-1})z + b_{k-1} - a_{k-1}}
\end{aligned}
$$

is attained, so that

$$
\begin{aligned}
&2\mathrm{Re}(z-1)\partial_z \log(1 + z^{++}_{2k-1}\zeta) = \\
&\frac{|a+b|^2|z-1|^2 - 4|b|^2}{|(a+b)z+b-a|^2} - \frac{|(a-b)\zeta + a + b|^2|z-1|^2 - 4|a\zeta + b|^2}{|\zeta[(a-b)z+a+b] + (a+b)z + b - a|^2}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&-2\mathrm{Re}(z-1)\partial_z \log|Q(z_{2k-1}^{++},\zeta)|^2\\
&=\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)z+b-a|^2}-\frac{|(a+b)\zeta+b-a|^2(|z-1|^2-2)-2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(a+b)z+b-a]-[(a-b)z+a+b]|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)z+b-a|^2}-\frac{|(a+b)\overline{\zeta}+b-a|^2(|z-1|^2-2)-2^{k+1}(|\zeta-1|^2-2)}{|\overline{\zeta}[(a+b)z+b-a]-[(a-b)z+a+b]|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)z+b-a|^2}-\frac{|(a-b)\zeta+a+b|^2(|z-1|^2-2)+2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(a-b)z+a+b]+(a+b)z+b-a|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)z+b-a|^2}-\frac{|(a-b)\overline{\zeta}+a+b|^2(|z-1|^2-2)+2^{k+1}(|\zeta-1|^2-2)}{|\overline{\zeta}[(a-b)z+a+b]+(a+b)z+b-a|^2}.
\end{aligned}
$$

Finally, on basis of Lemma 2.11

$$
\begin{aligned}
&-(z-1)\partial_z \log(\zeta-\overline{z_{2k}^{++}})\\
&=\frac{(-2)^{k+1}(z-1)}{\zeta[(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}]-[(\overline{a_{k-1}}-\overline{b_{k-1}})z+\overline{a_{k-1}}+\overline{b_{k-1}}]}\\
&\times\frac{1}{(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}}\\
&=\frac{2(\overline{b_{k-1}}\zeta-\overline{a_{k-1}})}{\zeta[(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}]-[(\overline{a_{k-1}}-\overline{b_{k-1}})z+\overline{a_{k-1}}+\overline{b_{k-1}}]}\\
&-\frac{2\overline{b_{k-1}}}{(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}}
\end{aligned}
$$

and thus as before

$$
\begin{aligned}
&-2\mathrm{Re}(z-1)\partial_z \log(\zeta-\overline{z_{2k}^{++}})=\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(\overline{a}+\overline{b})z+\overline{b}-\overline{a}|^2}\\
&-\frac{|(\overline{a}+\overline{b})\zeta+\overline{b}-\overline{a}|^2(|z-1|^2-2)-2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(\overline{a}+\overline{b})z+\overline{b}-\overline{a}]-[(\overline{a}-\overline{b})z+\overline{a}+\overline{b}]|^2}.
\end{aligned}
$$

And

$$
\begin{aligned}
&(z-1)\partial_z \log(1+\overline{z_{2k}^{++}}\zeta)\\
&=\frac{2(\overline{a_{k-1}}\zeta+\overline{b_{k-1}})}{\zeta[(\overline{a_{k-1}}-\overline{b_{k-1}})z+\overline{a_{k-1}}+\overline{b_{k-1}}]+(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}}\\
&-\frac{2\overline{b_{k-1}}}{(\overline{a_{k-1}}+\overline{b_{k-1}})z+\overline{b_{k-1}}-\overline{a_{k-1}}},
\end{aligned}
$$

so that (again working without indices)

$$
\begin{aligned}
&-2\mathrm{Re}(z-1)\partial_z \log(1+\overline{z_{2k}^{++}}\zeta) = \frac{|a+b|^2|z-1|^2-4|b|^2}{|(\overline{a}+\overline{b}z+\overline{b}-\overline{a})|^2}\\
&-\frac{|(\overline{a}-\overline{b})\zeta+\overline{a}+\overline{b}|^2|z-1|^2-4|\overline{a}\zeta+\overline{b}|^2}{|\zeta[(\overline{a}-\overline{b})z+\overline{a}+\overline{b}]+(\overline{a}+\overline{b})z+\overline{b}-\overline{a}|^2}\\
&=\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(\overline{a}+\overline{b})z+\overline{b}-\overline{a}|^2}\\
&-\frac{|(\overline{a}-\overline{b})\zeta+\overline{a}+\overline{b}|^2(|z-1|^2-2)+2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(\overline{a}-\overline{b})z+\overline{a}+\overline{b}]+(\overline{a}+\overline{b})z+\overline{b}-\overline{a}|^2}.
\end{aligned}
$$

Composing these terms shows

$$
\begin{aligned}
&-2\mathrm{Re}(z-1)\partial_z \log|Q(z_{2k}^{++},\zeta)|^2\\
&=\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)\overline{z}+b-a|^2}-\frac{|(\overline{a}+\overline{b})\zeta+\overline{b}-\overline{a}|^2(|z-1|^2-2)-2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(\overline{a}+\overline{b})z+\overline{b}-\overline{a}]-[(\overline{a}-\overline{b})z+\overline{a}+\overline{b}]|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)\overline{z}+b-a|^2}-\frac{|(a+b)\zeta+b-a|^2(|z-1|^2-2)-2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(a+b)\overline{z}+b-a]-[(a-b)\overline{z}+a+b]|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)\overline{z}+b-a|^2}-\frac{|(\overline{a}-\overline{b})\zeta+\overline{a}+\overline{b}|^2(|z-1|^2-2)+2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(\overline{a}-\overline{b})z+\overline{a}+\overline{b}]+(\overline{a}+\overline{b})z+\overline{b}-\overline{a}|^2}\\
&+\frac{|a+b|^2(|z-1|^2-2)-2^{k+1}}{|(a+b)\overline{z}+b-a|^2}-\frac{|(a-b)\zeta+a+b|^2(|z-1|^2-2)+2^{k+1}(|\zeta-1|^2-2)}{|\zeta[(a-b)\overline{z}+a+b]+(a+b)\overline{z}+b-a|^2}.
\end{aligned}
$$

In particular

$$
\begin{aligned}
&-2\mathrm{Re}(z-1)\partial_z \log|Q(z_0^{++},\zeta)|^2 \qquad (2.9)\\
&=\frac{(|z-1|^2-2)-2}{|z+1|^2}-\frac{|\zeta+1|^2(|z-1|^2-2)-2(|\zeta-1|^2-2)}{|\zeta(z+1)+z-1|^2}\\
&+\frac{(|z-1|^2-2)-2}{|z+1|^2}-\frac{|\zeta+1|^2(|z-1|^2-2)-2(|\zeta-1|^2-2)}{|\overline{\zeta}(z+1)+z-1|^2}\\
&+\frac{(|z-1|^2-2)-2}{|z+1|^2}-\frac{|\zeta-1|^2(|z-1|^2-2)+2(|\zeta-1|^2-2)}{|\zeta(z-1)-(z+1)|^2}\\
&+\frac{(|z-1|^2-2)-2}{|z+1|^2}-\frac{|\zeta-1|^2(|z-1|^2-2)+2(|\zeta-1|^2-2)}{|\overline{\zeta}(z-1)-(z+1)|^2}.
\end{aligned}
$$

The final calculation of the normal derivative of the Neumann function is achieved with the next lemma.

Lemma 2.12. *For* $|z-1|^2=2$ *the equations*

$$
\frac{2}{|(a+b)z+b-a|^2}=\frac{1}{|b\overline{z}+a|^2},
$$

$$
\frac{2}{|[(a+b)z+(b-a)]\zeta-[(a-b)z+a+b]|^2}=\frac{1}{|(b\overline{z}+a)\zeta+b-a\overline{z}|^2},
$$

$$
\frac{2}{|[(a-b)z+(a+b)]\zeta+(a+b)z+b-a|^2}=\frac{1}{|(a\overline{z}-b)\zeta+b\overline{z}+a|^2}
$$

hold. Moreover,

$$|(b\overline{z}+a)\zeta+b-a\overline{z}|^2=|(bz-a)\overline{\zeta}+az+b|^2,$$

$$|(a\overline{z}-b)\zeta+b\overline{z}+a|^2=|(az+b)\overline{\zeta}+a-bz|^2$$

are valid.

Proof. Multiplying numerators and denominators of the left-hand sides of the first three relations by $|z-1|^2$ and using $z(\overline{z}-1)=\overline{z}+1$ proves these. Looking at the respective differences for the last two equalities just $a\overline{b}+\overline{a}b=0$, see Lemma 2.5, has to be observed. □

Remark 2.13. Lemma 2.12 holds also if z is exchanged against $\overline{z}$ and independently by using $\overline{\zeta}$ in places of ζ. Altogether there are 8 appearances of the equations with z and ζ occurring commonly.

Lemma 2.14. *For* $\zeta\in\partial D,|\zeta-1|^2=2$

$$\begin{aligned}&\lim_{|z-1|^2\to 2}\Big[\sqrt{2}\partial_{\nu_z}N_1(z,\zeta)-2\Big[\frac{\zeta-1}{\zeta-z}+\frac{\overline{\zeta}-1}{\overline{\zeta}-z}-1\Big]\Big]\\&=-\sum_{k=0}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}z+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}z+a_{k-1}|^2}\Big]\\&-\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}\overline{z}+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}\overline{z}+a_{k-1}|^2}\Big],\end{aligned}$$

for $\zeta\in\overline{D},|\zeta-1|^2<2$

$$\begin{aligned}&\lim_{|z-1|^2\to 2}\sqrt{2}\partial_{\nu_z}N_1(z,\zeta)\\&=-\sum_{k=0}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}z+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}z+a_{k-1}|^2}\Big]\\&-\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}\overline{z}+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}\overline{z}+a_{k-1}|^2}\Big].\end{aligned}$$

Proof. Adding (2.8) and (2.9) shows for $|\zeta-1|^2=2$

$$\begin{aligned}&-2\mathrm{Re}(z-1)\partial_z\log|Q(z_0^+,\zeta)Q(z_0^{++},\zeta)|^2\\&=2\Big[\frac{\zeta-1}{\zeta-z}+\frac{\overline{\zeta}-1}{\overline{\zeta}-z}+1-\frac{|z-1|^2-2}{|\overline{\zeta}-z|^2}\\&-\frac{|\zeta|^2(|z-1|^2-2)}{|1+z\zeta|^2}-\frac{|\zeta|^2(|z-1|^2-2)}{|1+z\overline{\zeta}|^2}\Big]-4-\frac{8}{|z+1|^2}.\end{aligned}$$

When $|\zeta-1|^2=|z-1|^2=2$

$$\begin{aligned}&\sum_{k=1}^{\infty}2\mathrm{Re}(z-1)\partial_z\log|Q(z^+_{2k-1}Q(z^+_{2k}))Q(z^{++}_{2k-1}Q(z^+_{2k}))|^2\\&=\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}z+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}z+a_{k-1}|^2}\Big]\\&+\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}\overline{z}+b_{k-1}|^2}+\frac{2^{2k+2}}{|b_{k-1}\overline{z}+a_{k-1}|^2}\Big]\end{aligned}$$

is seen, where relations from Lemma 2.12 are used.

Further, for $|z-1|^2=2$ the terms with the factor $2(|\zeta-1|^2-2)$ as well in the sum

$$-2\mathrm{Re}(z-1)\partial_z\log|Q(z^+_{2k-1},\zeta)Q(z^{++}_{2k-1},\zeta)|^2$$

as in

$$-2\mathrm{Re}(z-1)\partial_z\log|Q(z^+_{2k},\zeta)Q(z^{++}_{2k},\zeta)|^2$$

cancel out. This is seen on basis of Lemma 2.12 from which also

$$-2\mathrm{Re}(z-1)\partial_z\log|Q(z^+_0,\zeta)Q(z^{++}_0,\zeta)|^2=-4-\frac{8}{|z+1|^2}$$

follows in this case. □

2.1.2. The right-hand boundary part $\{|z+1|^2=2\}$. The calculations for this boundary part are very similar to the ones from before and are based on the former derivatives. Just the results are listed, where again $a=a_{k-1},b=b_{k-1}$ is used.

$$\begin{aligned}&-2\mathrm{Re}(z+1)\partial_z\log|Q(z^+_0,\zeta)|^2=\Big[\frac{\zeta+1}{\zeta-z}+\overline{\frac{\zeta+1}{\zeta-z}}-1\Big]+\frac{|\zeta+1|^2-|z+1|^2}{|\overline{\zeta}-z|^2}\\&-\frac{|\zeta|^2(|z+1|^2-2)+|\zeta+1|^2-2}{|1+z\zeta|^2}-\frac{|\zeta|^2(|z+1|^2-2)+|\zeta+1|^2-2}{|1+z\overline{\zeta}|^2}-4,\\&-2\mathrm{Re}(z+1)\partial_z\log|Q(z^+_{2k-1},\zeta)|^2=4\frac{|a|^2(|z+1|^2-2)-2^k}{|az+b|^2}\\&+\frac{|a\zeta-b|^2(2-|z+1|^2)+(|\zeta+1|^2-2)2^k}{|(az+b)\zeta-bz+a|^2}\\&+\frac{|a\overline{\zeta}-b|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^k}{|(az+b)\overline{\zeta}-bz+a|^2}\\&+\frac{|b\zeta+a|^2(2-|z+1|^2)-(|\zeta+1|^2-2)2^k}{|(bz-a)\zeta+az+b|^2}\\&+\frac{|b\overline{\zeta}+a|^2(2-|z+1|^2)-(|\zeta+1|^2-2)2^k}{|(bz-a)\overline{\zeta}+az+b|^2},\end{aligned}$$

$$
\begin{aligned}
&- 2\mathrm{Re}(z+1)\partial_z \log|Q(z^{+}_{2k},\zeta)|^2 = 4\frac{|a|^2(|z+1|^2-2)-2^k}{|a\overline{z}+b|^2}\\
&+ \frac{|a\overline{\zeta}-b|^2(2-|z+1|^2)+(|\zeta+1|^2-2)2^k}{|(a\overline{z}+b)\overline{\zeta}-bz+a|^2}\\
&+ \frac{|a\zeta-b|^2(2-|z+1|^2)+(|\zeta+1|^2-2)2^k}{|(a\overline{z}+b)\zeta-b\overline{z}+a|^2}\\
&+ \frac{|b\overline{\zeta}+a|^2(2-|z+1|^2)-(|\zeta+1|^2-2)2^k}{|(b\overline{z}-a)\overline{\zeta}+a\overline{z}+b|^2}\\
&+ \frac{|b\zeta+a|^2(2-|z+1|^2)-(|\zeta+1|^2-2)2^k}{|(b\overline{z}-a)\zeta+a\overline{z}+b|^2},
\end{aligned}
$$

$$
\begin{aligned}
-2\mathrm{Re}(z+1)\partial_z \log|Q(z^{++}_{2k-1},\zeta)|^2 &= 4\frac{|a+b|^2(|z+1|^2-2)-2^{k+1}}{|(a+b)z+b-a|^2}\\
&-\frac{|(a+b)\zeta+b-a|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)z+b-a)\zeta+(b-a)z-(a+b)|^2}\\
&-\frac{|(a+b)\overline{\zeta}+b-a|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)z+b-a)\overline{\zeta}+(b-a)z-(a+b)|^2}\\
&+\frac{|(b-a)\zeta-a-b|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)+(a-b)z)\zeta+(a+b)z+b-a|^2}\\
&+\frac{|(b-a)\overline{\zeta}-a-b|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)+(a-b)z)\overline{\zeta}+(a+b)z+b-a|^2},\\
-2\mathrm{Re}(z+1)\partial_z \log|Q(z^{++}_{2k},\zeta)|^2 &= 4\frac{|a+b|^2(|z+1|^2-2)-2^{k+1}}{|(a+b)\overline{z}+b-a|^2}\\
&-\frac{|(a+b)\overline{\zeta}+b-a|^2(||z+1|^2-2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)\overline{z}+b-a)\overline{\zeta}+(b-a)\overline{z}-(a+b)|^2}\\
&-\frac{|(a+b)\zeta+b-a|^2(||z+1|^2-2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)\overline{z}+b-a)\zeta+(b-a)\overline{z}-(a+b)|^2}\\
&+\frac{|(b-a)\overline{\zeta}-a-b|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)+(a-b)\overline{z})\overline{\zeta}+(a+b)\overline{z}+b-a|^2},\\
&+\frac{|(b-a)\zeta-a-b|^2(2-||z+1|^2)+(|\zeta+1|^2-2)2^{k+1}}{|((a+b)+(a-b)\overline{z})\zeta+(a+b)\overline{z}+b-a|^2}.
\end{aligned}
$$

In particular

$$
\begin{aligned}
&- 2\mathrm{Re}(z+1)\partial_z \log|Q(z^{++}_0,\zeta)|^2 = 4\frac{|z+1|^2-4}{|z+1|^2}\\
&- \frac{|\zeta+1|^2(|z+1|^2-2)+2(|\zeta+1|^2-2)}{|(z+1)\zeta+z-1|^2}\\
&- \frac{|\zeta+1|^2(|z+1|^2-2)+2(|\zeta+1|^2-2)}{|(\overline{z}+1)\zeta+\overline{z}-1|^2}
\end{aligned}
$$

$$+\frac{|\zeta-1|^2(2-|z+1|^2)+2(|\zeta+1|^2-2)}{|(1-z)\zeta+z+1|^2}$$
$$+\frac{|\zeta-1|^2(2-|z+1|^2)+2(|\zeta+1|^2-2)}{|(1-\overline{z})\zeta+\overline{z}+1|^2}.$$

Hence, for $|\zeta+1|^2=2$

$$-2\mathrm{Re}(z+1)\partial_z\log|Q(z_0^+,\zeta)Q(z_0^{++},\zeta)|^2=2\Big[\frac{\zeta+1}{\zeta-z}+\frac{\overline{\zeta+1}}{\overline{\zeta-z}}-1\Big]+2\frac{2-|z+1|^2}{|\overline{\zeta}-z|^2}$$
$$-2\frac{|\zeta|^2(|z+1|^2-2)}{|1+z\zeta|^2}-2\frac{|\zeta|^2(|z+1|^2-2)}{|1+z\overline{\zeta}|^2}-4+4\frac{|z+1|^2-4}{|z+1|^2},$$

while for $|\zeta+1|^2<2=|z+1|^2$

$$-2\mathrm{Re}(z+1)\partial_z\log|Q(z_0^+,\zeta)Q(z_0^{++},\zeta)|^2=-8.$$

Obviously, in this latter case all terms with the factor $|\zeta+1|^2-2$ cancel out in the expressions for the $2\mathrm{Re}(z+1)\partial_z\log|Q(z_k^+,\zeta)|^2, 2\mathrm{Re}(z+1)\partial_z\log|Q(z_k^{++},\zeta)|^2, 0<k$, and this also holds for $2\mathrm{Re}(z+1)\partial_z\log|Q(z_0^+,\zeta)Q(z_0^{++},\zeta)|^2$. These insights are summarized in a lemma.

Lemma 2.15. *For* $\zeta\in\partial D, |\zeta+1|^2=2$

$$\lim_{|z+1|^2\to2}\Big[\sqrt{2}\partial_{\nu_z}N_1(z,\zeta)-2\Big[\frac{\zeta+1}{\zeta-z}+\frac{\overline{\zeta}+1}{\overline{\zeta-z}}-1\Big]\Big]$$
$$=-\sum_{k=0}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}z+b_{k-1}|^2}+\frac{2^{2k+2}}{|a_{k-1}z-b_{k-1}|^2}\Big]$$
$$-\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}\overline{z}+b_{k-1}|^2}+\frac{2^{2k+2}}{|a_{k-1}\overline{z}-b_{k-1}|^2}\Big],$$

for $\zeta\in\overline{D}, |\zeta+1|^2<2$

$$\lim_{|z+1|^2\to2}\sqrt{2}\partial_{\nu_z}N_1(z,\zeta)$$
$$=-\sum_{k=0}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}z+b_{k-1}|^2}+\frac{2^{2k+2}}{|a_{k-1}z-b_{k-1}|^2}\Big]$$
$$-\sum_{k=1}^{\infty}\Big[\frac{2^{2k+2}}{|a_{k-1}\overline{z}+b_{k-1}|^2}+\frac{2^{2k+2}}{|a_{k-1}\overline{z}-b_{k-1}|^2}\Big].$$

2.1.3. The upper boundary part $\{|z-im_1|^2=2\}$. Here the inner normal derivative to this circle

$$\sqrt{2}\partial_{\nu_z}N_1(z,\zeta)=2\mathrm{Re}(z-im_1)\partial_z\sum_{k=0}^{\infty}\log|Q(z_k^+,\zeta)Q(z_k^{++},\zeta)|^2$$

is to be evaluated, where

$$\sum_{k=0}^{\infty} \log |Q(z_k^+, \zeta)Q(z_k^{++}, \zeta)|^2 = \log |Q(z_0^+, \zeta)Q(z_0^{++}, \zeta)|^2$$
$$+ \sum_{k=1}^{\infty} \log |Q(z_{2k}^+, \zeta)Q(z_{2k-1}^+, \zeta)Q(z_{2k}^{++}, \zeta)Q(z_{2k-1}^{++}, \zeta)|^2$$

is used. The procedure is as before, leading to

$$\begin{aligned}
&2\mathrm{Re}(z - im_1)\partial_z \log Q(z_0^+, \zeta) \\
&= \Big[\frac{z - im_1}{z - \zeta} + \frac{\overline{z - im_1}}{\overline{z - \zeta}} - 1\Big] - \frac{|\zeta + im_1|^2 - |z - im_1|^2}{|\overline{\zeta} - z|^2} \\
&+ \frac{|\zeta|^2(|z - im_1|^2 - 2) - (|\zeta - im_1|^2 - 2)}{|1 + z\zeta|^2} \\
&+ \frac{|\zeta|^2(|z - im_1|^2 - 2) - (|\zeta + im_1|^2 - 2)}{|1 + z\overline{\zeta}|^2} + 4,
\end{aligned}$$

and using coefficients relations from Lemma 2.4 and Remark 2.6

$$\begin{aligned}
&2\mathrm{Re}(z - im_1)\partial_z \log |Q(z_{2k}^+, \zeta)|^2 = 4\frac{|a_k|^2 - |a_{k-1}|^2|z - im_1|^2}{|a_{k-1}\overline{z} + b_{k-1}|^2} \\
&- \frac{|a_k\overline{\zeta} - b_k|^2 - |a_{k-1}\overline{\zeta} - b_{k-1}|^2(|z - im_1|^2)}{|a_{k-1}(1 + \overline{z\zeta}) + b_{k-1}(\overline{\zeta - z})|^2} \\
&- \frac{|a_k\zeta - b_k|^2 - |a_{k-1}\zeta - b_{k-1}|^2(|z - im_1|^2)}{|a_{k-1}(1 + \overline{z}\zeta) + b_{k-1}(\zeta - \overline{z})|^2} \\
&- \frac{|b_k\overline{\zeta} + a_k|^2 - |b_{k-1}\overline{\zeta} + a_{k-1}|^2(|z - im_1|^2)}{|b_{k-1}(1 + \overline{z\zeta}) - a_{k-1}(\overline{\zeta - z})|^2} \\
&- \frac{|b_k\zeta + a_k|^2 - |b_{k-1}\zeta + a_{k-1}|^2(|z - im_1|^2)}{|b_{k-1}(1 + \overline{z}\zeta) - a_{k-1}(\zeta - \overline{z})|^2}.
\end{aligned}$$

The remaining terms for the normal derivative of N_1 are

$$\begin{aligned}
&2\mathrm{Re}(z - im_1)\partial_z \log |Q(z_{2k-1}^+, \zeta)|^2 = 4\frac{4|a_{k-2}|^2 - |a_{k-1}|^2|z - im_1|^2}{|a_{k-1}z + b_{k-1}|^2} \\
&- \frac{4|a_{k-2}\zeta - b_{k-2}|^2 - |a_{k-1}\zeta - b_{k-1}|^2|z - im_1|^2}{|a_{k-1}(1 + z\zeta) + b_{k-1}(\zeta - z)|^2} \\
&- \frac{4|a_{k-2}\overline{\zeta} - b_{k-2}|^2 - |a_{k-1}\overline{\zeta} - b_{k-1}|^2|z - im_1|^2}{|a_{k-1}(1 + z\overline{\zeta}) + b_{k-1}(\overline{\zeta} - z)|^2} \\
&- \frac{4|b_{k-2}\zeta + a_{k-2}|^2 - |b_{k-1}\zeta + a_{k-1}|^2|z - im_1|^2}{|b_{k-1}(1 + z\zeta) - a_{k-1}(\zeta - z)|^2} \\
&- \frac{4|b_{k-2}\overline{\zeta} + a_{k-2}|^2 - |b_{k-1}\overline{\zeta} + a_{k-1}|^2|z - im_1|^2}{|b_{k-1}(1 + z\overline{\zeta}) - a_{k-1}(\overline{\zeta} - z)|^2},
\end{aligned}$$

$$
\begin{aligned}
&2\mathrm{Re}(z-im_1)\partial_z \log|Q(z_{2k}^{++},\zeta)|^2 = 4\frac{|a_k+b_k|^2-|a_{k-1}+b_{k-1}|^2|z-im_1|^2}{|(a_{k-1}+b_{k-1})\overline{z}+b_{k-1}-a_{k-1}|^2} \\
&-\frac{|(a_k+b_k)\overline{\zeta}+b_k-a_k|^2-|(a_{k-1}+b_{k-1})\overline{\zeta}+b_{k-1}-a_{k-1}|^2|z-im_1|^2)}{|(a_{k-1}+b_{k-1})(\overline{z\zeta}-1)+(b_{k-1}-a_{k-1})(\overline{\zeta+z})|^2} \\
&-\frac{|(a_k+b_k)\zeta+b_k-a_k|^2-|(a_{k-1}+b_{k-1})\zeta+b_{k-1}-a_{k-1}|^2|z-im_1|^2)}{|(a_{k-1}+b_{k-1})(\overline{z}\zeta-1)+(b_{k-1}-a_{k-1})(\zeta+\overline{z})|^2} \\
&-\frac{|(a_k-b_k)\overline{\zeta}+a_k+b_k|^2-|(a_{k-1}-b_{k-1})\overline{\zeta}+a_{k-1}+b_{k-1}|^2|z-im_1|^2)}{|(a_{k-1}-b_{k-1})(\overline{z\zeta}-1)+(a_{k-1}+b_{k-1})(\overline{\zeta+z})|^2} \\
&-\frac{|(a_k-b_k)\zeta+a_k+b_k|^2-|(a_{k-1}-b_{k-1})\zeta+a_{k-1}+b_{k-1}|^2|z-im_1|^2)}{|(a_{k-1}-b_{k-1})(\overline{z}\zeta-1)+(a_{k-1}+b_{k-1})(\zeta+\overline{z})|^2},
\end{aligned}
$$

$$
\begin{aligned}
&2\mathrm{Re}(z-im_1)\partial_z \log|Q(z_{2k-1}^{++},\zeta)|^2 = 4\frac{4|a_{k-2}+b_{k-2}|^2-|a_{k-1}+b_{k-1}|^2|z-im_1|^2}{|(a_{k-1}+b_{k-1})z+b_{k-1}-a_{k-1}|^2} \\
&-\frac{4|(a_{k-2}+b_{k-2})\zeta+b_{k-2}-a_{k-2}|^2-|(a_{k-1}+b_{k-1})\zeta+b_{k-1}-a_{k-1}|^2|z-im_1|^2}{|(a_{k-1}+b_{k-1})(z\zeta-1)+(b_{k-1}-a_{k-1})(\zeta+z)|^2} \\
&-\frac{4|(a_{k-2}+b_{k-2})\overline{\zeta}+b_{k-2}-a_{k-2}|^2-|(a_{k-1}+b_{k-1})\overline{\zeta}+b_{k-1}-a_{k-1}|^2|z-im_1|^2}{|(a_{k-1}+b_{k-1})(z\overline{\zeta}-1)+(b_{k-1}-a_{k-1})(\overline{\zeta}+z)|^2} \\
&-\frac{4|(a_{k-2}-b_{k-2})\zeta+a_{k-2}+b_{k-2}|^2-|(a_{k-1}-b_{k-1})\zeta+a_{k-1}+b_{k-1}|^2|z-im_1|^2}{|(a_{k-1}-b_{k-1})(z\zeta-1)+(a_{k-1}+b_{k-1})(\zeta+z)|^2} \\
&-\frac{4|(a_{k-2}-b_{k-2})\overline{\zeta}+a_{k-2}+b_{k-2}|^2-|(a_{k-1}-b_{k-1})\overline{\zeta}+a_{k-1}+b_{k-1}|^2|z-im_1|^2}{|(a_{k-1}-b_{k-1})(z\overline{\zeta}-1)+(a_{k-1}+b_{k-1})(\overline{\zeta}+z)|^2}.
\end{aligned}
$$

In particular, recalling $a_{-1}=0, b_{-1}=1, a_0=1, b_0=-im_1$,

$$
\begin{aligned}
&2\mathrm{Re}(z-im_1)\partial_z \log|Q(z_0^{++},\zeta)|^2 = \frac{8}{|z+1|^2} \\
&-\frac{|\zeta-1+im_1(\zeta+1)|^2-2|\zeta+1|^2}{|z(\zeta+1)+\zeta-1|^2}-\frac{|\zeta-1-im_1(\zeta+1)|^2-2|\zeta+1|^2}{|\zeta-1+\overline{z}(\zeta+1)} \\
&-\frac{|\zeta+1-im_1(\zeta-1)|^2-2|\zeta-1|^2}{|\zeta+1-z(\zeta-1)|^2}-\frac{|\zeta+1+im_1(\zeta-1)|^2-2|\zeta-1|^2}{|\zeta+1-\overline{z}(\zeta-1)|^2} \\
&=\frac{8}{|z+1|^2}-2\frac{|\zeta+im_1|^2-2}{|z\zeta-1+\zeta+z|^2}-2\frac{|\zeta-im_1|^2-2}{|z\overline{\zeta}-1+\overline{\zeta}+z|^2} \\
&-2\frac{|\zeta+im_1|^2-2}{|z\zeta-1-(\zeta+z)|^2}-2\frac{|\zeta-im_1|^2-2}{|z\overline{\zeta}-1-(\overline{\zeta}+z)|^2}.
\end{aligned}
$$

Inserting $|z-im_1|^2=2$ in these expressions summing up as well

$$
2\mathrm{Re}(z-im_1)\partial_z\sum_{k=1}^{\infty}[\log|Q(z_{2k-1}^{+},\zeta)|^2|Q(z_{2k}^{+},\zeta)|^2]
$$

as

$$
2\mathrm{Re}(z-im_1)\partial_z\sum_{k=1}^{\infty}[\log|Q(z_{2k-1}^{++},\zeta)|^2|Q(z_{2k}^{++},\zeta)|^2]
$$

become telescope-sums. Obviously the terms can be paired. E.g. the first pair in the first listed sum is

$$\Sigma_1 = \sum_{k=1}^{\infty}\Big[4\frac{|a_k|^2 - |a_{k-1}|^2|z - im_1|^2}{|a_{k-1}\overline{z} + b_{k-1}|^2} + 4\frac{4|a_{k-2}|^2 - |a_{k-1}|^2|z - im_1|^2}{|a_{k-1}z + b_{k-1}|^2}\Big].$$

Using $\overline{z}(z - im_1) = -(1 + im_1 z)$ then, see Lemma 2.4,

$$\begin{aligned}\frac{|a_k|^2 - |a_{k-1}|^2|z - im_1|^2}{|a_{k-1}\overline{z} + b_{k-1}|^2} &= 2\frac{|a_k|^2 - 2|a_{k-1}|^2}{|(b_{k-1} - im_1 a_{k-1})z - a_{k-1} - im_1 b_{k-1}|^2}\\ &= 2\frac{|a_k|^2 - 2|a_{k-1}|^2}{|a_k z + b_k|^2}\end{aligned}$$

and hence

$$\begin{aligned}\Sigma_1 &= \sum_{k=1}^{\infty} 4\Big[2\frac{|a_k|^2 - 2|a_{k-1}|^2}{|a_k z + b_k|^2} - 2\frac{|a_{k-1}|^2 - 2|a_{k-2}|^2}{|a_{k-1}z + b_{k-1}|^2}\Big]\\ &= -8\frac{|a_0|^2 - 2|a_{-1}|^2}{|a_0 z + b_0|^2} = -4.\end{aligned}$$

In this way

$$\begin{aligned}&= 2\mathrm{Re}(z - im_1)\partial_z \sum_{k=1}^{\infty}[\log|Q(z_{2k-1}^{+}, \zeta)|^2|Q(z_{2k}^{+}, \zeta)|^2]\\ &= -4 + 2\frac{|\zeta + im_1|^2 - 2}{|1 + z\zeta - im_1(\zeta - z)|^2} + 2\frac{|\zeta - im_1|^2 - 2}{|1 + z\overline{\zeta} - im_1(\overline{\zeta} - z)|^2}\\ &+ 2\frac{|1 - im_1\zeta|^2 - 2|\zeta|^2}{|im_1(1 + z\zeta) + \zeta - z|^2} + 2\frac{|1 + im_1\zeta|^2 - 2|\zeta|^2}{|im_1(1 + z\overline{\zeta}) + \overline{\zeta} - z|^2}\\ &= -4 + \frac{|\zeta + im_1|^2 - 2}{|\zeta - \overline{z}|^2} + \frac{|\zeta - im_1|^2 - 2}{|\zeta - z|^2}\\ &+ \frac{|\zeta + im_1|^2 - 2}{|1 + \overline{z}\zeta|^2} + \frac{|\zeta - im_1|^2 - 2}{|1 + z\zeta|^2}\end{aligned}$$

is seen, coinciding with $-2\mathrm{Re}(z - im_1)\partial_z \log|Q(z_0^{+}, \zeta)|^2$ on $|z - im_1|^2 = 2$.

Accordingly for $|z - im_1|^2 = 2$

$$\begin{aligned}2\mathrm{Re}(z - im_1)\partial_z \sum_{k=1}^{\infty}[\log|Q(z_{2k-1}^{++}, \zeta)|^2|Q(z_{2k}^{++}, \zeta)|^2] &= \frac{-8}{|z + 1|^2}\\ &+ \frac{4(|\zeta - im_1|^2 - 2)}{|(1 - im_1)(z\zeta - 1) - (1 + im_1)(\zeta + z)|^2}\\ &+ \frac{4(|\zeta + im_1|^2 - 2)}{|(1 - im_1)(z\overline{\zeta} - 1) - (1 + im_1)(\overline{\zeta} + z)|^2}\end{aligned}$$

$$+\frac{4(|\zeta-im_1|^2-2)}{|(1+im_1)(z\zeta-1)+(1-im_1)(\zeta+z)|^2}$$
$$+\frac{4(|\zeta+im_1|^2-2)}{|(1+im_1)(z\overline{\zeta}-1)+(1-im_1)(\overline{\zeta}+z)|^2}$$
$$=\frac{-8}{|z+1|^2}+2\frac{|\zeta-im_1|^2-2}{|\overline{z}\zeta-1+\zeta+\overline{z}|^2}+2\frac{|\zeta+im_1|^2-2}{|z\zeta-1+\zeta+z|^2}$$
$$+2\frac{|\zeta-im_1|^2-2}{|\overline{z}\zeta-1-(\zeta+\overline{z})|^2}+2\frac{|\zeta+im_1|^2-2}{|z\zeta-1-(\zeta+z)|^2}$$

is shown, and thus being equal to $-2\mathrm{Re}(z-im_1)\partial_z\log|Q(z_0^{++},\zeta)|^2$.

Altogether $\partial_{\nu_z}N_1(z,\zeta)=0$ for $|z-im_1|^2=2<|\zeta-im_1|^2$ is deduced.

Next the normal derivative with respect to z of $N_1(z,\zeta)$ has to be calculated for $|\zeta-im_1|^2=2$. Using $\zeta(\overline{\zeta}+im_1)=im_1\overline{\zeta}-1$ together with the coefficients relations from Lemma 4 the formulas

$$|a_k\overline{\zeta}-b_k|^2=2|a_{k-1}\zeta-b_{k-1}|^2,\ \ |b_k\overline{\zeta}+a_k|^2=2|b_{k-1}\zeta+a_{k-1}|^2,$$
$$2|a_{k-1}(1+z\zeta)+b_{k-1}(\zeta-z)|^2=|a_k(1+z\overline{\zeta})+b_k(\overline{\zeta}-z)|^2,$$
$$2|b_{k-1}(1+z\zeta)-a_{k-1}(\zeta-z)|^2=|b_k(1+z\overline{\zeta})-a_k(\overline{\zeta}-z)|^2$$

are used. Similarly, for $|z-im_1|^2=2$ also

$$2|a_{k-1}\overline{z}-b_{k-1}|^2=|a_kz+b_k|^2,$$
$$2|a_{k-1}(1+\overline{z}\zeta)+b_{k-1}(\zeta-\overline{z})|^2=|a_k(1+z\zeta)+b_k(\zeta-z)|^2,$$
$$2|b_{k-1}(1+\overline{z}\zeta)-a_{k-1}(\zeta-\overline{z})|^2=|b_k(1+z\zeta)-a_k(\zeta-z)|^2$$

are needed. Thus for $1\le k$

$$2\mathrm{Re}(z-im_1)\partial_z\log|Q(z_{2k}^{+},\zeta)|^2=8\frac{|a_k|^2-2|a_{k-1}|^2}{|a_kz-b_k|^2}$$
$$-\frac{2|a_{k-1}\zeta-b_{k-1}|^2-4|a_{k-2}\zeta-b_{k-2}|^2}{|a_{k-1}(1+z\zeta)+b_{k-1}(\zeta-z)|^2}-\frac{2|a_k\zeta-b_k|^2-4|a_{k-1}\zeta-b_{k-1}|^2}{|a_k(1+z\zeta)+b_k(\zeta-z)|^2}$$
$$-\frac{2|b_{k-1}\zeta+a_{k-1}|^2-4|b_{k-2}\zeta+a_{k-2}|^2}{|b_{k-1}(1+z\zeta)-a_{k-1}(\zeta-z)|^2}-\frac{2|b_k\zeta+a_k|^2-4|b_{k-1}\zeta\zeta+a_{k-1}|^2}{|b_k(1+z\zeta)-a_k(\zeta-z)|^2}$$

and

$$2\mathrm{Re}(z-im_1)\partial_z\log|Q(z_{2k-1}^{+},\zeta)|^2=8\frac{2|a_{k-2}|^2-|a_{k-1}|^2}{|a_{k-1}z-b_{k-1}|^2}$$
$$+\frac{2|a_{k-1}\zeta-b_{k-1}|^2-4|a_{k-2}\zeta-b_{k-2}|^2}{|a_{k-1}(1+z\zeta)+b_{k-1}(\zeta-z)|^2}+\frac{2|a_{k-2}\zeta-b_{k-2}|^2-4|a_{k-3}\zeta-b_{k-3}|^2}{|a_{k-2}(1+z\zeta)+b_{k-2}(\zeta-z)|^2}$$
$$+\frac{2|b_{k-1}\zeta+a_{k-1}|^2-4|b_{k-2}\zeta+a_{k-2}|^2}{|b_{k-1}(1+z\zeta)-a_{k-1}(\zeta-z)|^2}+\frac{2|b_{k-2}\zeta+a_{k-2}|^2-4|b_{k-3}\zeta+a_{k-3}|^2}{|b_{k-2}(1+z\zeta)-a_{k-2}(\zeta-z)|^2}$$

is seen. Adding both terms and summing up the resulting telescope sum reduces to

$$\sum_{k=1}^{\infty} 2\text{Re}(z-im_1)\partial_z \log |Q(z_{2k-1}^+,\zeta)Q(z_{2k}^+,\zeta)|^2 = -8\frac{|a_0|^2-2|a_{-1}|^2}{|a_0z+b_0|^2}$$
$$+\frac{2|a_{-1}\zeta-b_{-1}|^2-4|a_{-2}\zeta-b_{-2}|^2}{|a_{-1}(1+z\zeta)+b_{-1}(\zeta-z)|^2}+\frac{2|a_0\zeta-b_0|^2-4|a_{-1}\zeta-b_{-1}|^2}{|a_0(1+z\zeta)+b_0(\zeta-z)|^2}$$
$$+\frac{2|b_{-1}\zeta+a_{-1}|^2-4|b_{-2}\zeta+a_{-2}|^2}{|b_{-1}(1+z\zeta)-a_{-1}(\zeta-z)|^2}+\frac{2|b_0\zeta+a_0|^2-4|b_{-1}\zeta+a_{-1}|^2}{|b_0(1+z\zeta)-a_0(\zeta-z)|^2}$$
$$=-4+\frac{2-|\zeta-im_1|^2}{|\zeta-z|^2}+\frac{2|\zeta+im_1|^2-4}{1+z\zeta-im_1(\zeta-z)|^2}$$
$$+\frac{2|\zeta|^2-|im_1\zeta+1|^2}{|1+z\zeta|^2}+\frac{2|1-im_1\zeta|^2-4|\zeta|^2}{|im_1(1+z\zeta\zeta)+\zeta-z|^2}.$$

Here the coefficients $a_0 = b_{-1} = 1, b_0 = -im_1, a_{-1} = 0, 2a_{-2} = 1, 2b_{-2} = im_1$ were inserted, see Lemma 2.4 and Remark 2.6. The second term in the resulting sum becomes singular when both z and ζ are located at the upper boundary part from D. In fact this expression is

$$\frac{2-|\zeta-im_1|^2}{|\zeta-z|^2}=-\Big[\frac{\zeta-im_1}{\zeta-z}+\frac{\overline{\zeta-im_1}}{\overline{\zeta-z}}-1\Big].$$

The two remaining terms can be simplified by observing for $|z-im_1|^2=2$

$$|1+z\zeta-im_1(\zeta-z)|^2=2|\zeta-\overline{z}|^2,\ \ |im_1(1+z\zeta)+\zeta-z|^2=2|1+\overline{z}\zeta|^2,$$

and also

$$|1+im_1\zeta|^2-2|\zeta|^2=|\zeta-im_1|^2-2,\ \ |1-im_1\zeta|^2-2|\zeta|^2=|\zeta+im_1|^2-2.$$

As for $|z-im_1|^2=2$

$$2\text{Re}(z-im_1)\partial_z\log|Q(z_0^+,\zeta)|^2=4-\Big[\frac{\zeta-im_1}{\zeta-z}+\frac{\overline{\zeta-im_1}}{\overline{\zeta-z}}-1\Big]$$
$$-\frac{|\zeta+im_1|^2-2}{|\zeta-\overline{z}|^2}-\frac{|\zeta-im_1|^2-2}{|1+z\zeta|^2}-\frac{|\zeta+im_1|^2-2}{|1+\overline{z}\zeta|^2},$$

then for $|z-im_1|^2=2$ as long as $|\zeta-im_1|^2>2$

$$2\text{Re}(z-im_1)\partial_z\log|Q(z_0^+,\zeta)|^2+\sum_{k=1}^{\infty}2\text{Re}(z-im_1)\partial_z\log|Q(z_{2k-1}^+,\zeta)Q(z_{2k}^+,\zeta)|^2=0.$$

If $|\zeta-im_1|^2=2$ then

$$2\text{Re}(z-im_1)\partial_z\log|Q(z_0^+,\zeta)|^2+\sum_{k=1}^{\infty}2\text{Re}(z-im_1)\partial_z\log|Q(z_{2k-1}^+,\zeta)Q(z_{2k}^+,\zeta)|^2$$
$$=-2\Big[\frac{\zeta-im_1}{\zeta-z}+\frac{\overline{\zeta-im_1}}{\overline{\zeta-z}}-1\Big].$$

In the same manner

$$\sum_{k=1}^{\infty} 2\mathrm{Re}(z-im_1)\partial_z \log|Q(z_{2k-1}^{++},\zeta)Q(z_{2k}^{++},\zeta)|^2 = -8\frac{|1-im_1|^2-2}{|(1-im_1)z-(1+im_1)|^2}$$
$$+2\frac{|(1-im_1)\zeta-(1+im_1)|^2-2|\zeta+1|^2}{|(1-im_1)(z\zeta-1)-(1+im_1)(\zeta+z)|^2}$$
$$+2\frac{|(1-im_1)\overline{\zeta}-(1+im_1)|^2-2|\zeta-1|^2}{|(1-im_1)(z\overline{\zeta}-1)-(1+im_1)(\overline{\zeta}+z)|^2}$$
$$+2\frac{|(1+im_1)\zeta+1-im_1|^2-2|\zeta-1|^2}{|(1+im_)(z\zeta-1)+(1-im_1)(\zeta+z)|^2}$$
$$+2\frac{|(1+im_1)\overline{\zeta}+1-im_1|^2-2|\zeta-1|^2}{|(1+im_)(z\overline{\zeta}-1)+(1-im_1)(\overline{\zeta}+z)|^2}$$
$$=-\frac{8}{|z+1|^2}+2\frac{|\zeta-im_1|^2-2}{|\overline{z}\zeta-1+\zeta+\overline{z}|^2}+2\frac{|\zeta+im_1|^2-2}{|\zeta z-1+\zeta+z|^2}$$
$$+2\frac{|\zeta-im_1|^2-2}{|\overline{z}\zeta-1-\zeta-\overline{z}|^2}+2\frac{|\zeta+im_1|^2-2}{|z\zeta-1-\zeta-z|^2}$$
$$=-2\mathrm{Re}(z-im1)\partial_z\log|Q(z_0^{++},\zeta)|^2.$$

Hence, for $|\zeta-im_1|^2=2$

$$\lim_{|z-im_1|^2\to 2}\Big[\partial_z N_1(z,\zeta)+\sqrt{2}\Big(\frac{\zeta-im_1}{\zeta-z}+\frac{\overline{\zeta-im_1}}{\overline{\zeta-z}}-1\Big)\Big]=0$$

and for $\zeta\in\overline{D}$ but $|\zeta-im_1|^2>2$

$$\lim_{|z-im_1|^2\to 2}\partial_z N_1(z,\zeta)=0$$

follow.

2.1.4. The lower boundary part $\{z-\overline{z}=0\}$. Here

$$\partial_{\nu_z}N_1(z,\zeta)=-i(\partial_z-\partial_{\overline{z}})N_1(z,\zeta)=2\mathrm{Im}\partial_z N_1(z,\zeta)$$

is relevant. The same procedure reveals

$$-2\mathrm{Im}\partial_z\log Q(z_0^{+},\zeta)=-i\frac{z-\overline{z}}{|\zeta-z|^2}+i\frac{\zeta-\overline{\zeta}}{|\zeta-z|^2}-i\frac{z-\overline{z}+(\zeta-\overline{\zeta})}{|\overline{\zeta}-z|^2}$$
$$+i\frac{|\zeta|^2(z-\overline{z})-(\zeta-\overline{\zeta})}{|1+z\zeta|^2}+i\frac{|\zeta|^2(z-\overline{z})+(\zeta-\overline{\zeta})}{|1+z\overline{\zeta}|^2},$$

$$-2\mathrm{Im}\partial_z\log\overline{Q(z_0^{++},\zeta)}=4i\frac{z-\overline{z}}{|z+1|^2}$$
$$-i\frac{|\zeta+1|^2(z-\overline{z})+2(\zeta-\overline{\zeta})}{|z\zeta-1+\zeta+z|^2}-i\frac{|\zeta+1|^2(z-\overline{z})-2(\zeta-\overline{\zeta})}{|\overline{z}\zeta-1+\zeta+\overline{z}|^2}$$
$$-i\frac{|\zeta-1|^2(z-\overline{z})+2(\zeta-\overline{\zeta})}{|1-z\zeta+\zeta+z|^2}-i\frac{|\zeta-1|^2(z-\overline{z})-2(\zeta-\overline{\zeta})}{|1-\overline{z}\zeta+\zeta+\overline{z}|^2},$$

and with $a = a_{k-1}, b = b_{k-1}$

$$
\begin{aligned}
-2\mathrm{Im}\partial_z \log Q(z_{2k}^{+},\zeta) &= 4i\frac{|a|^2(z-\overline{z}) + a\overline{b} - \overline{a}b}{|a\overline{z}+b|^2} \\
&- i\frac{|a\overline{\zeta}-b|^2(z-\overline{z}) - (|a|^2+|b|^2)(\zeta-\overline{\zeta}) + (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|a(1+\overline{z\zeta}) + b(\overline{\zeta - z})|^2} \\
&- i\frac{|a\zeta-b|^2(z-\overline{z}) + (|a|^2+|b|^2)(\zeta-\overline{\zeta}) + (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|a(1+\overline{z}\zeta) + b(\zeta - \overline{z})|^2} \\
&- i\frac{|b\overline{\zeta}+a|^2(z-\overline{z}) - (|a|^2+|b|^2)(\zeta-\overline{\zeta}) + (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|b(1+\overline{z\zeta}) - a(\overline{\zeta - z})|^2} \\
&- i\frac{|b\zeta+a|^2(z-\overline{z}) + (|a|^2+|b|^2)(\zeta-\overline{\zeta}) + (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|b(1+\overline{z}\zeta) - a(\zeta - \overline{z})|^2},
\end{aligned}
$$

$$
\begin{aligned}
-2\mathrm{Im}\partial_z \log \overline{Q(z_{2k-1}^{+},\zeta)} &= 4i\frac{|a|^2(z-\overline{z}) - (a\overline{b} - \overline{a}b)}{|az+b|^2} \\
&- i\frac{|a\zeta-b|^2(z-\overline{z}) - (|a|^2+|b|^2)(\zeta-\overline{\zeta}) - (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|a(1+z\zeta) + b(\zeta - z)|^2} \\
&- i\frac{|a\overline{\zeta}-b|^2(z-\overline{z}) + (|a|^2+|b|^2)(\zeta-\overline{\zeta}) - (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|a(1+z\overline{\zeta}) + b(\overline{\zeta} - z)|^2} \\
&- i\frac{|b\zeta+a|^2(z-\overline{z}) - (|a|^2+|b|^2)(\zeta-\overline{\zeta}) - (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|b(1+z\zeta) - a(\zeta - z)|^2} \\
&- i\frac{|b\overline{\zeta}+a|^2(z-\overline{z}) + (|a|^2+|b|^2)(\zeta-\overline{\zeta}) - (a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|b(1+z\overline{\zeta}) - a(\overline{\zeta} - z)|^2},
\end{aligned}
$$

$$
\begin{aligned}
&-2\mathrm{Im}\partial_z \log \overline{Q(z_{2k}^{++},\zeta)} = 4i\frac{|a+b|^2(z-\overline{z}) - 2(a\overline{b} - \overline{a}b)}{|(a+b)\overline{z}+b-a|^2} \\
&- i\frac{|(a+b)\overline{\zeta}+b-a|^2(z-\overline{z}) + 2(|a|^2+|b|^2)(\zeta-\overline{\zeta}) - 2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a+b)(\overline{z\zeta}-1) + (b-a)(\overline{\zeta + z})|^2} \\
&- i\frac{|(a+b)\zeta+b-a|^2(z-\overline{z}) - 2(|a|^2+|b|^2)(\zeta-\overline{\zeta}) - 2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a+b)(\overline{z}\zeta-1) + (b-a)(\zeta + \overline{z})|^2} \\
&- i\frac{|(a-b)\overline{\zeta}+a+b|^2(z-\overline{z}) + 2(|a|^2+|b|^2)(\zeta-\overline{\zeta}) - 2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a-b)(\overline{z\zeta}-1) + (a+b)(\overline{\zeta + z})|^2} \\
&- i\frac{|(a-b)\zeta+a+b|^2(z-\overline{z}) - 2(|a|^2+|b|^2)(\zeta-\overline{\zeta}) - 2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a-b)(\overline{z}\zeta-1) + (a+b)(\zeta + \overline{z})|^2},
\end{aligned}
$$

$$
\begin{aligned}
&- 2\mathrm{Im}\partial_z \log Q(z^{++}_{2k-1},\zeta) = 4i\frac{|a+b|^2(z-\overline{z})+2(a\overline{b}-\overline{a}b)}{|(a+b)z+b-a|^2}\\
&- i\frac{|(a+b)\zeta+b-a|^2(z-\overline{z})-2(|a|^2+|b|^2)(\zeta-\overline{\zeta})+2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a+b)(z\zeta-1)+(b-a)(\zeta+z)|^2}\\
&- i\frac{|(a+b)\overline{\zeta}+b-a|^2(z-\overline{z})+2(|a|^2+|b|^2)(\zeta-\overline{\zeta})+2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a+b)(z\overline{\zeta}-1)+(b-a)(\overline{\zeta}+z)|^2}\\
&- i\frac{|(a-b)\zeta+a+b|^2(z-\overline{z})+2(|a|^2+|b|^2)(\zeta-\overline{\zeta})+2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a-b)(z\zeta-1)+(a+b)(\zeta+z)|^2}\\
&- i\frac{|(a-b)\overline{\zeta}+a+b|^2(z-\overline{z})-2(|a|^2+|b|^2)(\zeta-\overline{\zeta})+2(a\overline{b}-\overline{a}b)(1+|\zeta|^2)}{|(a-b)(z\overline{\zeta}-1)+(a+b)(\overline{\zeta}+z)|^2}.
\end{aligned}
$$

Obviously, for $\zeta-\overline{\zeta}=0$

$$\lim_{z-\overline{z}\to 0}\Big[\partial_{\nu_z}N_1(z,\zeta)+i\frac{z-\overline{z}}{|\zeta-z|^2}\Big]=0.$$

For $z=\overline{z}$ and $\zeta\in\overline{D}$ but $\zeta\neq\overline{\zeta}$

$$\partial_{\nu_z}N_1(z,\zeta)=0,$$

as the respective terms as well from $Q(z^{+}_{2k},\zeta)$ and $Q(z^{+}_{2k-1},\zeta)$ as from $Q(z^{++}_{2k},\zeta)$ and $Q(z^{++}_{2k-1},\zeta)$ and those from $Q(z^{+}_{0},\zeta)$ and from $Q(z^{++}_{0},\zeta)$ cancel one another.

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