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# DIVISION BY ZERO CALCULUS IN ORDINARY DIFFERENTIAL EQUATIONS

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ABSTRACT. The aim of this note is to exemplify the importance of the division by zero calculus in the solutions of the Initial Value Problems. Several times we find Ordinary Differential Equations undefined to some initial conditions. Here, using the division by zero calculus, we can solve several ODEs to some new initial conditions.

## 1. Introduction

Brahmagupta was the first mathematician who gave the identity $0/0 = 0$ (century VII). During long time this equality was not accepted, however in 1924 this problem was discussed again by H. G. Romig [6]. Since then several works were published (see [1]-[5], [7], [8], [9] and references therein). The aim of this work is to exemplify how can we solve Initial Value Problems of Ordinary Differential Equations when we meet constants determined by the division by zero calculus.

## 2. Examples and open problems

On the first example we will consider a first order ordinary differential equation. Let $k > 1$. Consider the initial value problem

$$y'(x) = y^k(x), \tag{2.1}$$
$$y(x_0) = 0. \tag{2.2}$$

Then we have a general solution

$$y(x) = (1-k)^{1/(1-k)} \left(\frac{1}{x+C}\right)^{1/(k-1)}$$

of the equation (2.1) with a constant $C$. Since the initial condition is $y(x_0) = 0$, by division by zero calculus, we must have

$$0 = \frac{1}{0} \Leftrightarrow C = -x_0.$$

So, the solution of the (2.1)-(2.2) is

$$y(x) = (1-k)^{1/(1-k)} \left(\frac{1}{x-x_0}\right)^{1/(k-1)}.$$

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Here note that for any positive real number $p$, for the function

$$f_p(x) = \frac{1}{x^p}, \quad x > 0,$$

we have

$$f_p(0) = 0$$

([10]).

The coefficient

$$(1-k)^{1/(1-k)}$$

is not real valued.

For $k > 0$, in the initial problem for the differential equation

$$y' = \frac{-k}{x - x_0} y$$

with the initial condition

$$y(x_0) = 0,$$

we have the solution

$$y = \frac{1}{(x - x_0)^k}.$$

Meanwhile, for $k > 0$, in the initial problem for the differential equation

$$y' = -k y^{(1/k)+1}$$

with the initial condition

$$y(x_0) = 0,$$

we have the solution

$$y = \frac{1}{(x - x_0)^k}.$$

Let us consider now a second order differential equation. Consider the ordinary differential equation

$$x^2 y''(x) - 2xy'(x) + 2y(x) = \ln x. \tag{2.3}$$

The solution of (2.3) is given by

$$y(x) = c_1 x^2 + c_2 x + \frac{3}{4} + \frac{\ln x}{2}.$$

So, by division by zero calculus, we get

$$\begin{aligned} y(0) &= 3/4 \\ y'(0) &= c_2 \\ y''(0) &= 2c_1 \\ y^{(n)}(0) &= 0, \ \forall n \geq 3. \end{aligned}$$

It is thus evident that the division by zero calculus allows us to determine particular solutions that until now were not possible. However, much remains to be done in order to answer many open questions. Let's look at the next example. Consider the ordinary differential equation

$$x^3y'' + x^2y' - xy = \frac{1}{x}. \tag{2.4}$$

The solution of (2.4) is given by

$$y(x) = c_1x + \frac{c_2}{x} + \frac{1}{3x^2}.$$

So, by division by zero calculus, we get

$$\begin{aligned} y(0) &= 0 \\ y'(0) &= c_1 \\ y^{(n)}(0) &= 0, \ \forall n \geq 2. \end{aligned}$$

Here it was not possible to find a value to the constant $c_2$. However, with the analytic extension of the function $y(x)$, by the residue theorem we can determine the constant $c_2$, naturally.

**Open Problems**:

(1) How to find a constant when the derivatives of the solution is zero, as in the last Example?
(2) How can we resolve the lack of continuity regarding the initial conditions?
(3) The theory of the differential equations need to be review to include the division by zero calculus?

The author of this brief note hopes to help the scientific community to think about the problem of dividing by zero calculus and how many open problems can now be solved and how many others must be rethought. See also [1, 5] and [9] for many problems.

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