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KLEIN-GORDON'S EQUATION FOR MAGNONS WITHOUT NON-IDEAL EFFECT ON SPATIAL SEPARATION OF SPIN WAVES

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Abstract. In this paper we analyze a Klein-Gordon equation, which arises in the context of the physics of antiferromagnetic magnons, in order to determine whether a beam of linearly polarized spin wave could be separated into two secondary beams, in each of which one can find both a left and a right circularly polarized mode, which can be called (similar to what we know for the Stern-Gerlach effect) a non-ideal version of such a separation. The analysis carried out seeks to identify a new element to evaluate the Wang-Bao-Cao-Yan proposal of a "Magnonic Stern-Gerlach effect". In a complementary way, it is shown that the method adopted in our analysis is adequate to reveal incomplete spatial separation in the non-ideal Stern-Gerlach effect.

1. Introduction

In [1], in the context of the physics of antiferromagnetic magnons, Wang-Bao-Cao-Yan proposed a physical analogy based on the behavior of a linearly polarized beam of spin waves, which upon entering a region where an interaction of the type Dzyaloshinskii-Moriya[1], two secondary beams are produced, in each of which the spin waves present separately right and left circular polarization, which is considered an (ideal) Stern-Gerlach effect with magnonic origin. This proposal may receive additional support or will have to be reassessed as new information emerges. In this article we seek to identify, or rule out, the existence of a solution of the Klein-Gordon magnonic equation [1] compatible with a "non-ideal" analogue of the Stern-Gerlach effect.

As we know, the Stern-Gerlach effect [2], [3] is presented in Quantum Mechanics texts through its simplest version [4],[5],[6],[7],[8],[9], that is, when in each secondary beam we have only electrons in the same spin state: *up* in one beam and *down* in the other (in the case of an incident beam formed by Silver atoms). This situation corresponds to what can be called the "ideal Stern-Gerlach effect", or, simply, the "Stern-Gerlach effect". According to [10], however, in a real Stern-Gerlach experiment, both spin *up* electrons and spin *up* electrons can be found in each secondary beam, a manifestation that is known as the "non-ideal

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[1]Which effect could be interpreted, within the analogy established in [1], as "analogous" to that produced by the interaction between the magnetic spin moment of an electron (of a silver atom) with the magnetic field generated by a Stern-Gerlach device.

Stern-Gerlach effect" [11]. In papers [14]-[18]. The questions of exact and approximate solutions of the ill-posed Cauchy problem for various factorizations of the Helmholtz equations are studied. Such problems arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.).

1.1. Klein-Gordon magnonic equation for antiferromagnets. The relevant equation for our analysis is of the Klein-Gordon type [1],

$$\gamma^2\lambda\left[\left(\frac{A}{2}\nabla^2 - k\right)\Psi \;-\; iD\,\sigma_3\,\partial_2\Psi\right] \;=\; \partial_t^2\Psi \tag{1.1}$$

where we have that,

$$\Psi = \begin{pmatrix} n_+ \\ n_- \end{pmatrix}, \qquad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.2}$$

and the numerical parameters appearing in (1.1) are defined in [1]. Making,

$$\zeta \equiv (\gamma^2\lambda A)/2, \qquad \varepsilon \equiv \gamma^2\lambda k$$

we rewrite equation (1.1) as,

$$\begin{aligned} \zeta\begin{pmatrix} \partial^2_{xx}+\partial^2_{yy} & 0 \\ 0 & \partial^2_{xx}+\partial^2_{yy} \end{pmatrix}\begin{pmatrix} n_+ \\ n_- \end{pmatrix} - \varepsilon\begin{pmatrix} n_+ \\ n_- \end{pmatrix} + \\ -\,iD\begin{pmatrix} \partial_y & 0 \\ 0 & -\partial_y \end{pmatrix}\begin{pmatrix} n_+ \\ n_- \end{pmatrix} = \partial_t^2\begin{pmatrix} n_+ \\ n_- \end{pmatrix} \end{aligned} \tag{1.3}$$

where, for simplicity, we use the notation: $\partial^2_{xx} \equiv \partial^2/\partial x^2$, etc.

2. Mathematical development

The matrix equation (1.3) can be written equivalently as two decoupled differential equations,

$$\zeta\left(\partial^2_{xx}+\partial^2_{yy}\right)n_+ \;-\; \varepsilon n_+ \;-\; iD\,\partial_y n_+ \;=\; \partial_t^2 n_+ \tag{2.1}$$

$$\zeta\left(\partial^2_{xx}+\partial^2_{yy}\right)n_- \;-\; \varepsilon n_- \;+\; iD\,\partial_y n_- \;=\; \partial_t^2 n_- \tag{2.2}$$

or, in a compact form,

$$\mathbf{\Omega} n_+ \;-\; \mathbf{B}n_+ \;=\; \mathbf{G}n_+, \tag{2.3}$$

$$\mathbf{\Omega} n_- \;+\; \mathbf{B}n_- \;=\; \mathbf{G}n_-, \tag{2.4}$$

where the symbols introduced correspond to differential operators and other simple expressions, as follows,

$$\mathbf{\Omega} \;\equiv\; \zeta\left(\partial^2_{xx}+\partial^2_{yy}\right) - \;\varepsilon, \qquad \mathbf{B} \;\equiv\; iD\partial_y, \qquad \mathbf{G} \;\equiv\; \partial_t^2, \tag{2.5}$$

In the next subsection, we present our calculation strategy.

2.1. Strategy to identify/discard an effect with incomplete spatial separation. In this section we present a procedure [12] that would allow us to identify (or rule out) an effect of incomplete[2] spatial separation of the polarization states of a spin wave incident on the secondary beams. The "non-ideal" situation, if it exists, would mathematically manifest itself as follows: each component of the spinor in the Klein-Gordon equation would be formed by two terms (functions), which would correspond to the "movement" in opposite directions[3]; that is, we would have left (and right) circular polarization states in each secondary beam, as occurs (with due adaptations) in the "non-ideal Stern-Gerlach effect", see Appendix. On the other hand, in the ideal situation, each component of the spinor would be formed by a single term.

In terms of specific symbols, in order for a solution of equations (2.3) and (2.4) to correspond (if any) to a situation where both beams contain left and right circular polarizations, we must verify that it is possible to write each spinor component $(\varphi_1,\ \varphi_2)^t$ as a sum of two independent contributions, χ_1 and χ_2, with one "moving" in the $+Y$ direction and the other in the opposite direction, $-Y$, of the reference considered in [1]. So, we must define, somehow conveniently, the functions χ_1 and χ_2 in terms of φ_1 and φ_2, therefore, inverting the mathematical relationships, we would obtain φ_1 and φ_2 in terms of χ_1 and χ_2.

2.1.1. *Implementation of the strategy.* As is considered in various calculation methods, we introduce two external elements (free parameters) into our problem and then assign them appropriate values. We start by multiplying (2.3) by a number "λ", which must be defined correctly, and we add the resulting expression with (2.4), so we get,

$$\mathbf{\Omega}\Big(n_- + \lambda n_+\Big) + \mathbf{B}\,\underbrace{\Big(n_- - \lambda n_+\Big)} = \mathbf{G}\Big(n_- + \lambda n_+\Big). \tag{2.6}$$

To continue with the second parameter, "m", not null, which, like λ, would have to take on appropriate values, we rewrite the term highlighted in (2.6) as follows,

$$n_- - \lambda n_+ \;=\; m\Big(n_- + \lambda n_+\Big), \tag{2.7}$$

Then we define,

$$\chi \;\equiv\; n_- + \lambda n_+.$$

So, from (2.6) and (2.7) we find the equation for the function χ,

$$\mathbf{\Omega}\chi + m\mathbf{B}\chi = \mathbf{G}\chi \tag{2.8}$$

and let's suppose that there are two non-zero numbers, λ_1 and λ_2, which allow us to generate two functions χ_1 and χ_2 as follows,

$$\chi_1 = n_- + \lambda_1 n_+ \tag{2.9}$$

$$\chi_2 = n_- + \lambda_2 n_+ \tag{2.10}$$

[2]What is called the "non-ideal effect", compared to what is the "ideal effect", of total or complete separation.

[3]Such "movement" is defined by the displacement of the projections of the secondary beams in the direction orthogonal to the propagation direction of the incident beam.

which, by imposition, separately verify equation (2.8) with the corresponding value of m,

$$\mathbf{\Omega}\chi_1 + m_1\mathbf{B}\chi_1 = \mathbf{G}\chi_1 \tag{2.11}$$

$$\mathbf{\Omega}\chi_2 + m_2\mathbf{B}\chi_2 = \mathbf{G}\chi_2 \tag{2.12}$$

Furthermore, by inverting relations (2.9) and (2.10), we have,

$$n_- = \Big(\lambda_2\ \chi_1 - \lambda_1\ \chi_2\Big)\Big/\Big(\lambda_2 - \lambda_1\Big) \tag{2.13}$$

$$n_+ = \Big(\chi_2 - \chi_1\Big)\Big/\Big(\lambda_2 - \lambda_1\Big) \tag{2.14}$$

(χ_1 and χ_2 would be the functions corresponding to the movement in opposite directions of the secondary beams that move away from each other; in this way, it would be expected that equations (2.11) and (2.12) could have solutions with these characteristics). From (2.14), we see the need to assume that λ_2 and λ_1 are different.

Since n_+, in (2.14) satisfies equation (2.3), we have that,

$$\Bigg(\mathbf{\Omega}\chi_2 + \mathbf{B}\chi_2 - \mathbf{G}\chi_2\Bigg) - \Bigg(\mathbf{\Omega}\chi_1 - \mathbf{B}\chi_1 - \mathbf{G}\chi_1\Bigg) = 0, \tag{2.15}$$

The functions χ_1 and χ_2, in (2.15), must satisfy equations (2.11) and (2.12), respectively, with adequate values, m_1 and m_2. We directly note that for (2.15) to be identically satisfied, one must choose, in (2.11), $m_1 = -1$, as well as choose, in (2.12), $m_2 = +1$.

On the other hand, substituting (2.13) in expression (2.4) we find,

$$\lambda_2\Bigg(\mathbf{\Omega}\chi_1 + \mathbf{B}\chi_1 - \mathbf{G}\chi_1\Bigg) - \lambda_1\Bigg(\mathbf{\Omega}\chi_2 + \mathbf{B}\chi_2 - \mathbf{G}\chi_2\Bigg) = 0 \tag{2.16}$$

Analogously to the previous development, in order for expression (2.16) to be satisfied identically, we must take $m_1 = +1$, as well as $m_2 = +1$, but these values, together, are incompatible with the values that these parameters must assume so that expression (2.15) can be satisfied (simultaneously). Consequently, there will be no solution consistent with a non-ideal effect. In the Appendix, we show that the strategy used here determines, in the Stern-Gerlach case, that the coupled equations for the Pauli spinor components lead to an incomplete spatial separation by spin states.

3. Conclusion

The Wang-Bao-Cao-Yan proposal of a Stern-Gerlach effect with antiferromagnetic magnons led us to consider the problem of establishing whether the Klein-Gordon magnonic equation is an effect of incomplete spatial separation by polarization states (right and left circular, both present in each of the secondary beams generated by a linearly polarized spin wave), are compatible or not, for which an adequate strategy was defined in subsection 2.1. We show that what corresponds is a situation of incompatibility, which characterizes a non-ideal spatial separation effect, a name that was taken from what is known as the non-ideal Stern-Gerlach effect. The Wang-Bao-Cao-Yan proposal is compatible with the ideal Stern-Gerlach effect, but not with the experimentally observed non-ideal Stern-Gerlach effect [10].

4. Appendix

4.1. Non-ideal Stern-Gerlach effect and incomplete separation by spin states. It can be mathematically illustrated that in the Stern-Gerlach effect the complete separation and the incomplete separation (by electron spin states) occur in orthogonal planes to each other. Next, we show how the non-ideal Stern-Gerlach effect is revealed using the method defined in Subsection 2.1

The "non-ideal" Stern-Gerlach effect results from coupling the equations for the Pauli spinor components, with the magnetic field gradient (α) acting as a coupling parameter. Let's first pay attention to the expression for the magnetic field,

$$\vec{B}(y,z) \;=\; (0, -\alpha y,\; B_0 + \alpha z), \tag{4.1}$$

commonly used in the literature on the Stern-Gerlach effect. Note that (4.1) checks for $\vec{\nabla} \cdot \vec{B} = 0$. Expression (4.1), however, is not completely satisfactory, as one would expect it to be invariant with respect to a Parity transformation, as it corresponds to pseudo-vector fields in the $(1+3)$-dimensional case.

Applying the Parity transformation in (4.1) we obtain the expression,

$$\vec{B}'(-y,-z) \;=\; -\,\vec{B}(y,z) \;+\; 2B_0\vec{k}, \tag{4.2}$$

from which we see that $\vec{B}$ would behave like a vector field if $B_0 = 0$, not being possible (consistently with a non-zero gradient) that (4.1) corresponds to a pseudo-vector field. The vector character of (4.2), with $B_0 = 0$, does not affect the invariance of Maxwell's equations (with $\rho = 0$, $\vec{j} = \vec{0}$ and $\partial\vec{E}/\partial t = 0$, corresponding to the Stern-Gerlach effect) in relation to a Parity transformation, as expected for purely electrical and magnetic interactions. For what follows, we keep the usual expression (4.1), taking $B_0 = 0$, that is, we represent $\vec{B}$ as if it were a vector.

4.2. Mathematical development. We consider the splitting of a bundle of Silver atoms, which initially moves along the positive direction of the X coordinate of the reference system considered, as in [1], which then splits into two secondary bundles in the horizontal plane $Z = 0$. The Pauli matrix equation, considering $B_0 = 0$, can be written as follows,

$$-\frac{\hbar^2}{2m}\begin{pmatrix} \partial^2_{xx} + \partial^2_{yy} & 0 \\ 0 & \partial^2_{xx} + \partial^2_{yy} \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} + \mu_B \begin{pmatrix} 0 & i\alpha y \\ -i\alpha y & 0 \end{pmatrix}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix} =$$

$$= i\hbar\frac{\partial}{\partial t}\begin{pmatrix} \eta_1 \\ \eta_2 \end{pmatrix}. \tag{4.3}$$

or, equivalently, as two coupled differential equations for the spinor components,

$$\tilde{\mathbf{\Omega}}\eta_1 \;+\; \tilde{\mathbf{G}}\eta_2 \;=\; \mathbf{D}\eta_1, \tag{4.4}$$

$$\tilde{\mathbf{\Omega}}\eta_2 \;-\; \tilde{\mathbf{G}}\eta_1 \;=\; \mathbf{D}\eta_2. \tag{4.5}$$

the symbols being introduced,

$$\tilde{\mathbf{\Omega}} \equiv -\frac{\hbar^2}{2m}\left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}\right), \ \tilde{\mathbf{G}} \equiv i\mu_B \alpha y, \ \mathbf{D} \equiv i\hbar\frac{\partial}{\partial t}. \tag{4.6}$$

Here we implement the strategy used in subsection 2.1.1; that is, we start by multiplying (4.5) by a suitable number "λ", to be defined, and we add it with (4.4), obtaining,

$$\tilde{\mathbf{\Omega}}\left(\eta_1 + \lambda\eta_2\right) + \tilde{\mathbf{G}}\left(\eta_2 - \lambda\eta_1\right) = \mathbf{D}\left(\eta_1 + \lambda\eta_2\right) \tag{4.7}$$

To continue, we introduce a second parameter, $m \neq 0$, which will also have to take a suitable value, so that we can rewrite the functions in the central term in (4.7) as follows,

$$\eta_2 - \lambda\eta_1 = m\left(\eta_1 + \lambda\eta_2\right). \tag{4.8}$$

With that, we define

$$\chi \equiv \eta_1 + \lambda\eta_2. \tag{4.9}$$

Then, we can identify, from (4.7), (4.8) and (4.9), the equation for the function χ,

$$\tilde{\mathbf{\Omega}}\chi + m\tilde{\mathbf{G}}\chi = \mathbf{D}\chi. \tag{4.10}$$

On the other hand, from (4.9), and for two different numbers λ_1 and λ_2 we impose that the functions χ_1 and χ_2 that follow,

$$\chi_1 = \eta_1 + \lambda_1\eta_2, \tag{4.11}$$

$$\chi_2 = \eta_1 + \lambda_2\eta_2, \tag{4.12}$$

correspond to the solutions of equation (4.10); that is, we can write,

$$\tilde{\mathbf{\Omega}}\chi_1 + m_1\tilde{\mathbf{G}}\chi_1 = \mathbf{D}\chi_1, \tag{4.13}$$

$$\tilde{\mathbf{\Omega}}\chi_2 + m_2\tilde{\mathbf{G}}\chi_2 = \mathbf{D}\chi_2. \tag{4.14}$$

Furthermore, inverting relations (4.11) and (4.12), we have,

$$\eta_1 = \left(\lambda_2\chi_1 - \lambda_1\chi_2\right)\Big/\left(\lambda_2 - \lambda_1\right) \tag{4.15}$$

$$\eta_2 = \left(-\chi_1 + \chi_2\right)\Big/\left(\lambda_2 - \lambda_1\right) \tag{4.16}$$

(we are assuming that $\lambda_2 - \lambda_1 \neq 0$). Since η_1, in (4.15), and η_2, in (4.16), should verify equation (4.4),

$$\tilde{\mathbf{\Omega}}\eta_1 + \tilde{\mathbf{G}}\eta_2 = \mathbf{D}\eta_1$$

Thus, we have, substituting the expressions (4.15) and (4.16) in this equation, and after sorting the terms, that,

$$\lambda_2\left(\tilde{\mathbf{\Omega}}\chi_1 - \frac{1}{\lambda_2}\tilde{\mathbf{G}}\chi_1 - \mathbf{D}\chi_1\right) - \lambda_1\left(\tilde{\mathbf{\Omega}}\chi_2 - \frac{1}{\lambda_1}\tilde{\mathbf{G}}\chi_2 - \mathbf{D}\chi_2\right) = 0. \tag{4.17}$$

As the functions χ_1 and χ_2, must verify equations (4.13) and (4.14), respectively, then, in order for expression (4.17) to be verified identically, we must assume the following relations,

$$m_1 = -\frac{1}{\lambda_2} \qquad e \qquad m_2 = -\frac{1}{\lambda_1}. \tag{4.18}$$

On the other hand, the functions η_1 and η_2, in (4.15) and (4.16), respectively, must satisfy equation (4.5); to mean,

$$\tilde{\mathbf{\Omega}}\eta_2 \; - \; \tilde{\mathbf{G}}\eta_1 \; = \; \mathbf{D}\eta_2.$$

So, substituting (4.15) and (4.16) into (4.5), and after sorting the terms, we have that,

$$-\left(\tilde{\mathbf{\Omega}}\chi_1 + \lambda_2\tilde{\mathbf{G}}\chi_1 - \mathbf{D}\chi_1\right) + \left(\tilde{\mathbf{\Omega}}\chi_2 + \lambda_1\tilde{\mathbf{G}}\chi_2 - \mathbf{D}\chi_2\right) = 0 \tag{4.19}$$

and as the functions χ_1 and χ_2, in the previous expression, verify equations (4.13) and (4.14), respectively, we see that for (4.19) to be verified identically, one must consider what,

$$m_1 = \lambda_2 \qquad e \qquad m_2 = \lambda_1 \tag{4.20}$$

Consequently, based on the mathematical consistency between equations (4.13), (4.14), (4.17) and (4.19), it must be fulfilled, from (4.18) and (4.20), that,

$$\lambda_2 = -\frac{1}{\lambda_2} \qquad e \qquad \lambda_1 = -\frac{1}{\lambda_1} \tag{4.21}$$

thus establishing their values, resulting in $\lambda_2 = \pm i$ and $\lambda_1 = \pm i$. It is convenient to choose $\lambda_2 = -i$; hence, $\lambda_1 = +i$, since these must be different, as previously assumed. Note the complete consistency of the values assigned: $\lambda_1 = i$, $\lambda_2 = -i$, $m_1 = -i$ and $m_2 = i$ with expression (4.8), which is checked for each pair formed by the values of λ and m.

With the values assigned to the parameters (which were previously free) we have that equations (4.13) and (4.14) are well defined, as follows,

$$\tilde{\mathbf{\Omega}}\chi_1 \; - \; i\tilde{\mathbf{G}}\chi_1 \; = \; \mathbf{D}\chi_1, \tag{4.22}$$

or explicitly,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi_1}{\partial x^2} + \frac{\partial^2\chi_1}{\partial y^2}\right) + \mu_{_B}\alpha y\chi_1 = i\hbar\frac{\partial\chi_1}{\partial t}. \tag{4.23}$$

And also,

$$\tilde{\mathbf{\Omega}}\chi_2 \; + \; i\tilde{\mathbf{G}}\chi_2 \; = \; \mathbf{D}\chi_1. \tag{4.24}$$

or explicitly,

$$-\frac{\hbar^2}{2m}\left(\frac{\partial^2\chi_2}{\partial x^2} + \frac{\partial^2\chi_2}{\partial y^2}\right) - \mu_{_B}\alpha y\chi_2 = i\hbar\frac{\partial\chi_2}{\partial t} \tag{4.25}$$

On the other hand, based on [13] equations (4.23) and (4.25), in the context of the Stern-Gerlach Effect, have the following exact solutions,

$$\chi_1(x,y,t) = Ai\left[a\left(y + (\xi_0/a) + bt^2\right)\right] \times \Theta\left[y - (\nu_0/a) + (\xi_0/a) + bt^2\right] \times$$

$$\times e^{i(cyt)} e^{(i/\hbar)[xp_x - \hbar\omega_+ t]} \tag{4.26}$$

$$\chi_2(x,y,t) = Ai\Big[a\Big(- y + (\xi_0/a) + bt^2\Big)\Big] \times \Theta\Big[- y - (\nu_0/a) + (\xi_0/a) + bt^2\Big]\times$$

$$\times e^{-i(cyt)} e^{(i/\hbar)[xp_x - \hbar\omega_- t]} \tag{4.27}$$

respectively, being,

$$a = \left(\frac{2m(\mu_B\alpha - 2mb)}{\hbar^2}\right)^{1/3}, \quad c = -\frac{2mb}{\hbar}, \quad b = \frac{\mu_B\alpha}{4m}$$

$$\hbar\omega_+ = \hbar\omega_- = \frac{p_x^2}{2m} - \frac{\xi_0}{2}\left(\frac{\mu_B^2\alpha^2\hbar^2}{m}\right)^{1/3} \tag{4.28}$$

Θ is the Heaviside function, ν_0 is the first zero of the Airy (Ai) function, and ξ_0 is the first maximum of Ai. Note that the argument of the function Ai, in (4.26), with $a > 0$, represents the "accelerated" part of the projection of the movement of a secondary beam in the $-Y$ direction and that the argument of the function Ai, in (4.27), represents the "accelerated" part of the projection of the movement of the second secondary beam in the opposite direction. Consequently, functions (4.15) and (4.16), components of the Pauli spinor, can be written separately as the sum of two contributions moving in opposite directions along the Y coordinate direction,

$$\eta_1(x,y,t) \;=\; \frac{1}{2}\Big(\chi_1(x,y,t) \;+\; \chi_2(x,y,t)\Big), \tag{4.29}$$

$$\eta_2(x,y,t) \;=\; \frac{1}{2i}\Big(\chi_1(x,y,t) \;-\; \chi_2(x,y,t)\Big), \tag{4.30}$$

which clearly constitute solutions that do not correspond to the "ideal Stern-Gerlach effect", but to a non-ideal effect, or incomplete separation by electronic spin states in the split beams, as observed experimentally according to [10].

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