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DEFORMED STOCKWELL TRANSFORM AND APPLICATIONS ON THE REPRODUCING KERNEL THEORY

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Abstract. In this paper, we study the generalized translation operator associated with the deformed Hankel transform on $\mathbb{R}$. Firstly, we prove the trigonometric form of the generalized translation operator. Next, we derive the positivity of this operator on a suitable space of even functions. Making use of the positivity of the generalized translation operator we introduce and we study the deformed Stockwell transform. Knowing the fact that the study of reproducing kernels theory are both theoretically interesting and practically useful, we investigate for this transform the general theory of reproducing kernels theory. In particular, we investigate some applications of the Tikhonov regularization for the generalized Sobolev spaces and we study some time-frequency concentration problems.

1. INTRODUCTION

In their seminal paper [1], Ben Saïd, Kobayashi and Ørsted have given a foundation of the deformation theory of the classical situation, by constructing a generalization $\mathcal{F}_{k,a}$ of the Fourier transform, and the holomorphic semigroup $\mathcal{I}_{k,a}(z)$ with infinitesimal generator

$$\mathcal{L}_{k,a} := ||x||^{2-a}\triangle_k - ||x||^a, \quad a > 0,$$

acting on a concrete Hilbert space deforming $L^2(\mathbb{R}^d)$. Here $\triangle_k$ is the Dunkl Laplacian see [11]. The (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ can be regarded as a two-parameter generalization of Howe's description of classical Fourier transform, where k is a multiplicity function for the Dunkl operators on $\mathbb{R}^d$ and $a > 0$ arises from the interpolation of the two Lie algebra $\mathcal{SL}(2,\mathbb{R})$ actions on the Weil representation of $Mp(d,\mathbb{R})$ and the minimal unitary representation of the $O(d + 1, 2)$.

The (k, a)-generalized Fourier transform $\mathcal{F}_{k,a}$ includes some prominent transforms on the Euclidean space $\mathbb{R}^d$:

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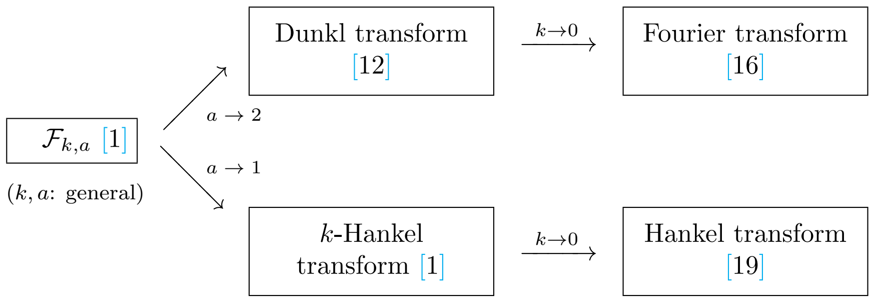
* This paper is dedicated to Professor Mohamed Sifi in occasion of his 65 birthday.



Recently, there is a growing interest to develop the analysis related to the (k, a)-generalised Fourier transform. Notably, maximal function and translation operator [2], uncertainty principles and Pitt inequalities [14,17], Fourier multipliers [18], wavelets multipliers [22,23], wavelet transform [24,25], localization operators [26], Gabor transform [27,28] and Hardy inequality [38] were explored by many researchers.

One of the aims of the Fourier transform, is the study of the time-frequency analysis. In the sixties the time-frequency analysis has emerged with the works of Gabor [13] who provided an interesting way to study the local frequency spectrum of signals by introducing many time-frequency representations, as, for instance, the short-time Fourier transform (STFT), the continuous wavelet transform or also the Wigner distribution where all of these representations have a same common point, that is the simultaneous representation of the space and the frequency variables in a same set called the time-frequency plane.

The major drawback of the short-time Fourier transform is the fixed width of the analysing window. Indeed, in many applications, the high frequency content of a signal is more time/space-localized than the low-frequency one. Removing of the rigidity of the window function is one of the motivations for continuous wavelet transform. Although, the wavelet transform captures more information than the short-time Fourier transform (STFT), however, it suffers from two apparent limitations: first, the detail measured by the wavelet transform is not directly analogs to the frequency, because the wavelet transform is essentially a time-scale transform with the inverse scale being interpreted as frequency; second, the phase-information is completely lost in the case of wavelet transform, because each wavelet component acts a local filter and the translation of the mother wavelet destroys the phase information with respect to the origin [39,41]. To circumvent these limitations, Stockwell et al. [37] introduced the notion of Stockwell transform as a bridge between the STFT and the wavelet transform. By adopting the progressive resolution of wavelets, the Stockwell transform is able to resolve a wider range of frequencies than the ordinary STFT and by using a Fourier-like basis and maintaining a phase of zero about the time $t = 0$, Fourier based analysis could be performed locally. This unique feature of the Stockwell transform makes it a highly valuable tool for signal processing and is one of the hottest research areas of the contemporary era. Indeed, the Stockwell transform has been successfully used to analyse signals in numerous applications, such as seismic recordings, ground vibrations, geophysics, medical imaging, hydrology, gravitational waves, power system analysis and many other areas. Finally, we note that many extensions of the Stockwell transform have been proposed in recent years. See, for example, [4, 6, 8, 10, 29, 34–36] and others.

In this paper, we consider the case $a = \frac{2}{n}$, $n \in \mathbb{N}$ and $d = 1$. We shall call the generalized Fourier transform $\mathcal{F}_{k,\frac{2}{n}}$ the deformed *Hankel transform* and we will denote it (simply) by $\mathcal{F}_{k,n}$.

The purpose of this document is threefold. On one hand, we want to study the generalized translation operator on the deformed Hankel setting. In particular, we prove its positivity on suitable space of functions. Profiting of this positivity the aim of the second part of this paper is to introduce the generalized Stockwell transform in the setting of the deformed Hankel transform and to study its harmonic analysis. Keeping in view the fact that the reproducing kernel theory for the deformed Stockwell transforms is yet to be investigated exclusively, our third endeavour is to study some problems of the reproducing kernel theory associated with this transform.

The main contributions of this article are as follows:

- To obtain the trigonometric formula for the generalized translation operator.
- To derive the positivity of the generalized translation operator on a suitable space of even functions.
- To introduce and to study the generalized Stockwell transform operator in the setting of the deformed Hankel transform.
- To introduce and to investigate the generalized Sobolev spaces $W^s_{k,n}(\mathbb{R})$ associated with the deformed Stockwell transform.
- To give some applications of the general theory of reproducing kernels to the Tikhonov regularization for the deformed Stockwell transform.
- To study some uncertainty principles for the deformed Stockwell transform.

The remainder of this paper is arranged as follows.

In Section 2, we recall the main results about the deformed Hankel transform. Section 3 is exclusively dedicated to study the generalized translation operator. In Section 4, we introduce and we study the deformed Stockwell transform. More precisely the inversion, Plancherel's and Lieb's formulas are established. Section 5 is devoted to introduce the generalized Sobolev spaces $W^s_{k,n}(\mathbb{R})$ associated with the deformed Stockwell transform. Afterwards, we give some applications of the general theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the deformed Stockwell transform on these generalized Sobolev spaces. Next, in Section 6 we establish the Heisenberg, Benedicks and Donoho-Stark's type uncertainty principles for the deformed Stockwell transform. Finally, the last section is devoted to study the Shapiro uncertainty principle for the deformed Stockwell transform.

2. PRELIMINARIES

This section gives an introduction to the harmonic analysis associated with the deformed Hankel transform. Main reference is [1].

Notation. Let us denote by

$C_b(\mathbb{R})$ the space of bounded continuous functions on $\mathbb{R}$.

$C_{b,e}(\mathbb{R})$ the space of even bounded continuous functions on $\mathbb{R}$.

For $p \in [1,\infty]$, p' denotes as in all that follows, the conjugate exponent of p.

$M_{k,n} := \frac{n^{\frac{n(2k-1)}{2}}}{2^{\frac{n(2k-1)+2}{2}}\Gamma(\frac{n(2k-1)+2}{2})}$.

$d\gamma_{k,n}(x) := M_{k,n}|x|^{\frac{(2k-2)n+2}{n}}dx$, $k \geq \frac{n-1}{n}$.

$L^p_{k,n}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, the space of measurable functions on $\mathbb{R}$ such that

$$\begin{aligned} ||f||_{L^p_{k,n}(\mathbb{R})} &:= \Big(\int_{\mathbb{R}} |f(x)|^p d\gamma_{k,n}(x)\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leqslant p < \infty, \\ ||f||_{L^\infty_{k,n}(\mathbb{R})} &:= \operatorname{ess}\sup_{x\in\mathbb{R}} |f(x)| < \infty. \end{aligned}$$

For $p = 2$, we provide this space with the scalar product

$$\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})} := \int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,n}(x).$$

For $k \geq \frac{n-1}{n}$, and $f \in L^1_{k,n}(\mathbb{R})$, the deformed Hankel transform is defined by

$$\mathcal{F}_{k,n}(f)(\lambda) = \int_{\mathbb{R}} f(x)B_{k,n}(\lambda, x)d\gamma_{k,n}(x), \quad \text{for all } \lambda \in \mathbb{R}, \tag{2.1}$$

where $B_{k,n}(\lambda, x)$ is the deformed Hankel kernel given by

$$B_{k,n}(\lambda, x) = \jmath_{nk-\frac{n}{2}}\big(n|\lambda x|^{\frac{1}{n}}\big) + \big(\frac{-in}{2}\big)^n \frac{\Gamma(nk - \frac{n}{2} + 1)}{\Gamma(nk + \frac{n}{2} + 1)}\lambda x \jmath_{nk+\frac{n}{2}}\big(n|\lambda x|^{\frac{1}{n}}\big). \tag{2.2}$$

Here

$$\jmath_\alpha(u) := \Gamma(\alpha + 1)\Big(\frac{u}{2}\Big)^{-\alpha} J_\alpha(u) = \Gamma(\alpha + 1)\sum_{m=0}^{\infty} \frac{(-1)^m}{m!\,\Gamma(\alpha + m + 1)}\Big(\frac{u}{2}\Big)^{2m} \tag{2.3}$$

denotes the normalized Bessel function of index α.

Next, we give some properties of the deformed Hankel kernel.

Proposition 2.1. i) *For $z, t \in \mathbb{R}$, we have*

$$B_{k,n}(z, t) = B_{k,n}(t, z), \quad B_{k,n}(z, 0) = 1, \quad \overline{B_{k,n}(z, t)} = B_{k,n}((-1)^n z, t)$$

and $B_{k,n}(\lambda z, t) = B_{k,n}(z, \lambda t)$ for all $\lambda \in \mathbb{R}$.

ii) *There exists a finite positive constant C only depends on n and k, such that for all $x, y \in \mathbb{R}$ we have*

$$|B_{k,n}(x, y)| \leqslant C.$$

Convention:([17]). We shall replace $B_{k,n}$ by the rescaled version $B_{k,n}/C$ but continue to use the same symbol $B_{k,n}$ and we obtain

$$\forall\, x, y \in \mathbb{R}, \quad |B_{k,n}(x, y)| \leqslant 1. \tag{2.4}$$

We note that the authors in [14] that conjectured (2.4) when $k \geq \frac{n-1}{n}$.

Remark 2.1. *(i) We note that the previous inequality implies that the deformed Hankel transform is bounded on the space* $L^1_{k,n}(\mathbb{R})$*, and we have*

$$||\mathcal{F}_{k,n}(f)||_{L^\infty_{k,n}(\mathbb{R})} \leqslant ||f||_{L^1_{k,n}(\mathbb{R})}, \tag{2.5}$$

for all f *in* $L^1_{k,n}(\mathbb{R})$*.*

(ii) The deformed Hankel transform $\mathcal{F}_{k,n}$ *provides a natural generalization of the Hankel transform. Indeed, if we set*

$$\begin{aligned} B^{even}_{k,n}(x,y) &= \tfrac{1}{2}\big(B_{k,n}(x,y) + B_{k,n}(x,-y)\big) \\ &= j_{nk-\frac{n}{2}}(n|xy|^{\frac{1}{n}}). \end{aligned}$$

Then, the deformed Hankel transform $\mathcal{F}_{k,n}$ *of an even function* f *on the real line specializes to a Hankel type transform on* $\mathbb{R}_+$*. In fact, when* $f(x) = F(|x|)$ *is an even function on* $\mathbb{R}$ *and belongs to* $L^1_{k,n}(\mathbb{R})$*, we have*

$$\forall \xi \in \mathbb{R},\ \mathcal{F}_{k,n}(f)(\xi) = \frac{(\frac{n}{2})^{(\frac{2nk-n}{2})}}{\Gamma(\frac{2nk+2-n}{2})} \int_0^\infty F(r) j_{\frac{2nk-n}{2}}\Big(n(r|\xi|)^{\frac{1}{n}}\Big) r^{\frac{2}{n}(\frac{2nk+2-n}{2})-1} dr. \tag{2.6}$$

Example 2.1. *The function* α_t*,* $t > 0$*, defined on* $\mathbb{R}$ *by*

$$\alpha_t(x) = \frac{1}{(2t)^{\frac{2nk+2-n}{2}}} e^{-\frac{n|x|^{\frac{2}{n}}}{4t}}, \tag{2.7}$$

satisfies

$$\forall \xi \in \mathbb{R},\ \mathcal{F}_{k,n}(\alpha_t)(\xi) = e^{-nt|\xi|^{\frac{2}{n}}}. \tag{2.8}$$

The authors in [1] have proved the following.

Proposition 2.2. i*) Plancherel's theorem for* $\mathcal{F}_{k,n}$*.*
The deformed Hankel transform $f \mapsto \mathcal{F}_{k,n}(f)$ *is an isometric isomorphism on* $L^2_{k,n}(\mathbb{R})$ *and we have*

$$\int_{\mathbb{R}} |f(x)|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\lambda)|^2 d\gamma_{k,n}(\lambda). \tag{2.9}$$

ii*) Parseval's formula for* $\mathcal{F}_{k,n}$*.*
For all f, g *in* $L^2_{k,n}(\mathbb{R})$ *we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)} d\gamma_{k,n}(x) = \int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\lambda)\overline{\mathcal{F}_{k,n}(g)(\lambda)} d\gamma_{k,n}(\lambda). \tag{2.10}$$

iii*) Inversion formula.*
The deformed Hankel transform is an involutive unitary operator on $L^1_{k,n}(\mathbb{R})$*, i.e., we have*

$$\mathcal{F}^{-1}_{k,n}(f)(x) = \mathcal{F}_{k,n}(f)((-1)^n x), \quad x \in \mathbb{R}. \tag{2.11}$$

Proposition 2.3. *Let* f *be in* $L^p_{k,n}(\mathbb{R})$*,* $p \in [1,2]$*. Then* $\mathcal{F}_{k,n}(f)$ *belongs to* $L^{p'}_{k,n}(\mathbb{R})$ *and we have*

$$\|\mathcal{F}_{k,n}(f)\|_{L^{p'}_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^p_{k,n}(\mathbb{R})}.$$

Definition 2.1. *Let U, V be two measurable subsets of $\mathbb{R}$. Then:*

(1) We say that the pair (U, V) is weakly annihilating, if $supp\, f \subset U$ and $supp\, \mathcal{F}_{k,n}(f) \subset V$ implies $f = 0$.

(2) We say that the pair (U, V) is strongly annihilating, if there exists a positive constant $C := C_{k,n}(U, V)$ such that for every function f in $L^2_{k,n}(\mathbb{R})$,

$$C\big(\|\mathcal{F}_{k,n}(f)\|^2_{L^2_{k,n}(V^c)} + ||f||^2_{L^2_{k,n}(U^c)}\big) \geq \|f\|^2_{L^2_{k,n}(\mathbb{R})}. \tag{2.12}$$

Here $A^c := \mathbb{R}\backslash A$ is the complement of A. The constant $C_{k,n}(U, V)$ will be called the annihilation constant of (U, V).

Now, we recall the following Benedicks-type uncertainty principle for the deformed Hankel transform proved by Johansen in [[17], Theorem 9.1].

Proposition 2.4. *Let U, V be two measurable subsets of $\mathbb{R}$ with*

$$\gamma_{k,n}(U) := \int_U d\gamma_{k,n}(x) < \infty \quad \text{and} \quad \gamma_{k,n}(V) := \int_V d\gamma_{k,n}(x) < \infty.$$

Then the pair (U, V) is a strongly annihilating pair.

3. GENERALIZED TRANSLATION OPERATOR

Definition 3.1. *Let $x \in \mathbb{R}$. We define the generalized translation operator $\tau^{k,n}_x$ on $L^2_{k,n}(\mathbb{R})$ by*

$$\mathcal{F}_{k,n}(\tau^{k,n}_x f) = \overline{B_{k,n}(.,x)}\mathcal{F}_{k,n}(f). \tag{3.1}$$

It is useful to have a class of functions in which (3.1) holds pointwise. One such class is given by the generalized Wigner space $\mathcal{W}_{k,n}(\mathbb{R})$ given by

$$\mathcal{W}_{k,n}(\mathbb{R}) := \Big\{f \in L^1_{k,n}(\mathbb{R}) : \mathcal{F}_{k,n}(f) \in L^1_{k,n}(\mathbb{R})\Big\}.$$

On the follow we give several properties of the generalized translation operator.

Proposition 3.1. *(i) Let f be in $L^2_{k,n}(\mathbb{R})$, we have*

$$\|\tau^{k,n}_x f\|_{L^2_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^2_{k,n}(\mathbb{R})}, \quad \forall x \in \mathbb{R}. \tag{3.2}$$

(ii) For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ we have

$$\tau^{k,n}_x f(y) = \int_{\mathbb{R}} B_{k,n}((-1)^n x, \xi)B_{k,n}((-1)^n y, \xi)\mathcal{F}_{k,n}(f)(\xi)d\gamma_{k,n}(\xi), \quad \forall x, y \in \mathbb{R}. \tag{3.3}$$

(iii) For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and for all $x, y \in \mathbb{R}$, we have

$$\tau^{k,n}_x f(y) = \tau^{k,n}_y (f)(x). \tag{3.4}$$

(iv) For all f in $\mathcal{W}_{k,n}(\mathbb{R})$ and $g \in L^1_{k,n}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, we have

$$\forall\, x \in \mathbb{R}, \quad \int_{\mathbb{R}} \tau^{k,n}_x f(y)g(y)d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\tau^{k,n}_{(-1)^n x} g(y)d\gamma_{k,n}(y). \tag{3.5}$$

Proof. For part (i), it is enough to use (3.1), Plancherel's identity (2.9) and relation (2.4). For part (ii) we use (3.1), inversion formula (2.11) and Proposition 2.1 i). For part (iii), it is enough to use the symmetry $B_{k,n}(x,y) = B_{k,n}(y,x)$. For part (iv), involving Parseval's formula (2.10), we get for all $x \in \mathbb{R}$

$$\begin{aligned}\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y) d\gamma_{k,n}(y) &= \int_{\mathbb{R}} \overline{B_{k,n}(x,\xi)} \mathcal{F}_{k,n}(f)(\xi) \overline{\mathcal{F}_{k,n}(g)(\xi)} d\gamma_{k,n}(\xi) \\ &= \int_{\mathbb{R}} f(y) \tau_{(-1)^n x}^{k,n} g(y) d\gamma_{k,n}(y).\end{aligned}$$

□

Recently the authors in [3] have given an explicit formula for the generalized translation operators given by the following.

Theorem 3.1. *Let $x \in \mathbb{R}$ and let $f \in C_b(\mathbb{R})$. For $k \geq \frac{n-1}{n}$, the generalized translation operator $\tau_x^{k,n}$ is given by*

$$\tau_x^{k,n} f(y) = \int_{\mathbb{R}} f(z) d\zeta_{x,y}^{k,n}(z), \tag{3.6}$$

here

$$d\zeta_{x,y}^{k,n}(z) = \begin{cases} \mathcal{K}_{k,n}(x,y,z) d\gamma_{k,n}(z), & \text{if} \quad xy \neq 0, \\ d\delta_x(z), & \text{if} \quad y = 0, \\ d\delta_y(z), & \text{if} \quad x = 0, \end{cases}$$

where $\mathcal{K}_{k,n}(x,y,.)$ is supported on the set

$$\Big\{ z \in \mathbb{R} : \quad \big|\, |x|^{\frac{1}{n}} - |y|^{\frac{1}{n}} \,\big| < |z|^{\frac{1}{n}} < |x|^{\frac{1}{n}} + |y|^{\frac{1}{n}} \Big\}$$

and is given by

$$\mathcal{K}_{k,n}(x,y,z) = K_{\mathrm{B}}^{nk-\frac{n}{2}}(|x|^{\frac{1}{n}}, |y|^{\frac{1}{n}}, |z|^{\frac{1}{n}}) \, \nabla_{k,n}(x,y,z), \tag{3.7}$$

where

$$\begin{aligned}\nabla_{k,n}(x,y,z) := \frac{M_{k,n}}{2n} \Big\{ 1 + (-1)^n \frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \big(\Delta(|x|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |z|^{\frac{2}{n}})\big) \\ + \frac{n!\mathrm{sgn}(xz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \big(\Delta(|z|^{\frac{2}{n}}, |x|^{\frac{2}{n}}, |y|^{\frac{2}{n}})\big) \\ + \frac{n!\mathrm{sgn}(yz)}{(2kn-n)_n} C_n^{nk-\frac{n}{2}} \big(\Delta(|z|^{\frac{2}{n}}, |y|^{\frac{2}{n}}, |x|^{\frac{2}{n}})\big) \Big\},\end{aligned} \tag{3.8}$$

$$\Delta(u,v,w) := \frac{1}{2\sqrt{uv}}(u+v-w), \qquad \textit{for } u,v,w \in \mathbb{R}_+^*, \tag{3.9}$$

$C_n^{nk-\frac{n}{2}}$ the Gegenbauer polynomials and $K_{\mathrm{B}}^{nk-\frac{n}{2}}$ is the positive kernel given by

$$K_{\mathrm{B}}^{nk-\frac{n}{2}}(u,v,w) = \frac{2\Gamma(nk-\frac{n}{2}+1)}{\Gamma(nk-\frac{n-1}{2})\Gamma(\frac{1}{2})} \frac{\Big\{ \big[(u+v)^2 - w^2\big] \big[w^2 - (u-v)^2\big] \Big\}^{nk-\frac{n+1}{2}}}{(2uvw)^{2nk-n}} \tag{3.10}$$

for $|u-v| < w < u+v$ and $K_B^{nk-\frac{n}{2}}(u,v,w) = 0$ elsewhere.

The explicit formula implies the boundedness of $\tau_y^{k,n} f$. More precisely, we have.

Proposition 3.2. *([3]) For all $f \in L^p_{k,n}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$, we have*

$$\forall\, x \in \mathbb{R}, \quad ||\tau^{k,n}_x f||_{L^p_{k,n}(\mathbb{R})} \leqslant 4||f||_{L^p_{k,n}(\mathbb{R})}. \tag{3.11}$$

On the follows we will prove the "trigonometric" form of the generalized translation operator.

Theorem 3.2. *For $f \in C_b(\mathbb{R})$ write $f = f_{\rm e} + f_{\rm o}$ as a sum of even and odd functions. Then*

$$\begin{aligned}
\tau^{k,n}_x f(y) &= \tfrac{M_{k,n}}{2n}\Big[\int_0^{\pi} f_{\rm e}\big(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\big)\Big\{1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\cos\phi\big)\Big\}\\
&+ f_{\rm o}\big(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\big)\Big\{\tfrac{n!\mathrm{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\tfrac{|x|^{\frac1n}-|y|^{\frac1n}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac1n}}\big)\\
&+ \tfrac{n!\mathrm{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\tfrac{|y|^{\frac1n}-|x|^{\frac1n}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac1n}}\big)\Big\}(\sin\phi)^{2nk-n}d\phi\Big],
\end{aligned}$$

where

$$\langle\!\langle x,y\rangle\!\rangle_{\phi,n} := \big(|x|^{\frac2n}+|y|^{\frac2n}-2|xy|^{\frac1n}\cos\phi\big)^{\frac{n}{2}}. \tag{3.12}$$

Proof. By (3.8), the even and odd parts of the function $\nabla_{k,n}(x,y,\cdot)$ are given respectively by

$$\begin{aligned}
\nabla_{k,n,{\rm e}}(x,y,z) &:= \frac{M_{k,n}}{2n}\Big\{1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|x|^{\frac2n},|y|^{\frac2n},|z|^{\frac2n})\big)\Big\},\\
\nabla_{k,n,{\rm o}}(x,y,z) &:= \frac{M_{k,n}}{2n}\Big\{\frac{n!\mathrm{sgn}(xz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac2n},|x|^{\frac2n},|y|^{\frac2n})\big)\\
&+ \frac{n!\mathrm{sgn}(yz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac2n},|y|^{\frac2n},|x|^{\frac2n})\big)\Big\}.
\end{aligned}$$

Hence, equation (3.6) turns into

$$\begin{aligned}
\tau^{k,n}_x f(y) &= 2\int_0^{\infty} f_{\rm e}(z)K_{\rm B}^{nk-\frac{n}{2}}(|x|^{\frac1n},|y|^{\frac1n},|z|^{\frac1n})\,\nabla_{k,n,{\rm e}}(x,y,z)\,d\gamma_{k,n}(z)\\
&+ 2\int_0^{\infty} f_{\rm o}(z)K_{\rm B}^{nk-\frac{n}{2}}(|x|^{\frac1n},|y|^{\frac1n},|z|^{\frac1n})\,\nabla_{k,n,{\rm o}}(x,y,z)\,d\gamma_{k,n}(z).
\end{aligned}$$

For

$$\big|\,|x|^{\frac1n}-|y|^{\frac1n}\,\big| < |z|^{\frac1n} < |x|^{\frac1n}+|y|^{\frac1n},$$

we may substitute

$$\cos\phi := \frac{|x|^{\frac2n}+|y|^{\frac2n}-|z|^{\frac2n}}{2|xy|^{\frac1n}} = \Delta(|x|^{\frac2n},|y|^{\frac2n},|z|^{\frac2n}) \tag{3.13}$$

with $\phi \in [0,\pi]$. Moreover involving (3.13) and (3.12), we get

$$\begin{aligned}
\Delta(|z|^{\frac2n},|x|^{\frac2n},|y|^{\frac2n}) &= \tfrac{|z|^{\frac2n}+|x|^{\frac2n}-|y|^{\frac2n}}{2|x|^{\frac1n}|z|^{\frac1n}}\\
&= \tfrac{|x|^{\frac1n}-|y|^{\frac1n}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac1n}}
\end{aligned}$$

Thus, for $z>0$, we infer

$$\frac{n!\mathrm{sgn}(xz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac{2}{n}},|x|^{\frac{2}{n}},|y|^{\frac{2}{n}})\big)=\frac{n!\mathrm{sgn}(x)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\Big(\frac{|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\Big).$$

Similarly we prove that

$$\frac{n!\mathrm{sgn}(yz)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\Delta(|z|^{\frac{2}{n}},|y|^{\frac{2}{n}},|x|^{\frac{2}{n}})\big)=\frac{n!\mathrm{sgn}(y)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\Big(\frac{|y|^{\frac{1}{n}}-|x|^{\frac{1}{n}}\cos\phi}{\langle\!\langle x,y\rangle\!\rangle_{\phi,n}^{\frac{1}{n}}}\Big).$$

Thus, the generalized translation operator takes the desired form. □

Below we will study the positivity of the generalized translation operator on even functions in $\mathcal{W}_{k,n}(\mathbb{R})$, which is far from being obvious. This result will be crucial for the rest of the paper. To do so, we will give an explicit expression of the translation operator acting on such functions.

Corollary 3.1. *For all f in $C_{b,e}(\mathbb{R})$, we have*

$$\tau_x^{k,n}f(y)=\frac{M_{k,n}}{2n}\int_0^\pi f\big(\langle\!\langle x,y\rangle\!\rangle_{\phi,n}\big)\mathcal{N}_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n}d\phi,$$

where

$$\mathcal{N}_{k,n}(x,y,\phi):=1+(-1)^n\frac{n!\mathrm{sgn}(xy)}{(2kn-n)_n}C_n^{nk-\frac{n}{2}}\big(\cos\phi\big).$$

Involving the previous Corollary we infer the following

Lemma 3.1. *For every $\lambda>0$ and for every $x\in\mathbb{R}$, we have*

$$\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y)=\frac{M_{k,n}}{2n}e^{-\lambda\big(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\big)}V_{k,n}(\lambda;x,y),$$

where

$$V_{k,n}(\lambda;x,y):=\int_0^\pi e^{2\lambda|xy|^{\frac{1}{n}}\cos\phi}\mathcal{N}_{k,n}(x,y,\phi)(\sin\phi)^{2nk-n}d\phi.$$

Remark 3.1. *Involving the previous lemma, the properties of the Gegenbauer polynomials and by simple calculations we infer that there exist a positive constant $C(k,n)$ such that*

$$|\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y)|\leqslant C(k,n)e^{-\lambda\big(|x|^{\frac{1}{n}}-|y|^{\frac{1}{n}}\big)^2}.$$

Now, let us go back to the properties of the generalized translation operator.

Proposition 3.3. *Let f be an nonnegative even function of $\mathcal{W}_{k,n}(\mathbb{R})$. Then*
(i) For any $x\in\mathbb{R}$, we have $\tau_x^{k,n}f\geq 0$.
(ii) For every $x\in\mathbb{R}$, we have $\tau_x^{k,n}f\in L^1_{k,n}(\mathbb{R})$ and

$$\int_{\mathbb{R}}\tau_x^{k,n}f(y)d\gamma_{k,n}(y)=\int_{\mathbb{R}}f(y)d\gamma_{k,n}(y).\tag{3.14}$$

Proof. Using the explicit expression of the generalized translation operator given in Corollary 3.1, the properties of the Gegenbauer polynomials and by simple calculations we prove the first statement. To prove (ii), let us substitute $g(y)$ by $e^{-\lambda|y|^{\frac{2}{n}}}$ in the relation (3.5). Thus by Lemma 3.1, we get

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}} d\gamma_{k,n}(y) = \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda;(-1)^n x, y) d\gamma_{k,n}(y). \tag{3.15}$$

Using the fact that $\tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}} \geq 0$, and the monotone convergence theorem we get

$$\lim_{\lambda\to 0} \int_{\mathbb{R}} \tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{2}{n}}} d\gamma_{k,n}(y) = \int_{\mathbb{R}} \tau_x^{k,n} f(y) d\gamma_{k,n}(y). \tag{3.16}$$

Now we will estimate

$$\lim_{\lambda\to 0} \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda;(-1)^n x, y) d\gamma_{k,n}(y).$$

In view of the upper estimate for $\tau_x^{k,n}(e^{-\lambda|.|^{\frac{2}{n}}})(y)$ in Remark 3.1, the dominated convergence theorem gives

$$\lim_{\lambda\to 0} \frac{M_{k,n}}{2n} \int_{\mathbb{R}} f(y) e^{-\lambda\left(|x|^{\frac{2}{n}}+|y|^{\frac{2}{n}}\right)} V_{k,n}(\lambda;(-1)^n x, y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) d\gamma_{k,n}(y). \tag{3.17}$$

Combining the relations (3.15), (3.16) and (3.17) we infer the desired result. □

We close this paragraph by giving the second main result of this section.

Theorem 3.3. *Let $L^p_{k,n,e}(\mathbb{R})$ be the space of even functions in $L^p_{k,n}(\mathbb{R})$.*

(i) Let $f \in L^1_{k,n,e}(\mathbb{R})$ be bounded and nonnegative. Then we have

$$\forall\, x \in \mathbb{R}, \quad \tau_x^{k,n} f \geq 0, \quad \tau_x^{k,n} f \in L^1_{k,n}(\mathbb{R})$$

and

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) d\gamma_{k,n}(y). \tag{3.18}$$

(ii) The generalized translation operator initially defined on $L^1_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$ can be extended to all $L^p_{k,n,e}(\mathbb{R})$, $1 \leqslant p \leqslant \infty$ and for all f in $L^p_{k,n,e}(\mathbb{R})$, we have

$$\forall\, x \in \mathbb{R}, \quad ||\tau_x^{k,n} f||_{L^p_{k,n}(\mathbb{R})} \leqslant ||f||_{L^p_{k,n}(\mathbb{R})}. \tag{3.19}$$

(iii) For every $f \in L^1_{k,n}(\mathbb{R})$ we have

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y) d\gamma_{k,n}(y). \tag{3.20}$$

By means of the generalized translation operator, we define the generalized convolution product of two suitable functions f and g by

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y) g(y) d\gamma_{k,n}(y). \tag{3.21}$$

Remark 3.2. *(i) It is clear that this convolution product is both commutative and associative.*

(ii) This convolution structure carries a new commutative signed hypergroup in the sense of [30] or [31]. The concept of signed hypergroups generalizes the

hypergroup axiomatics in several fact, mainly in abandoning positivity and support continuity of the convolution.

We close the notion of the generalized convolution product by giving the following results.

Proposition 3.4. (See [3]) *The following statements hold true.*

(i) *Let* $f \in L^2_{k,n}(\mathbb{R})$ *and* $g \in L^1_{k,n}(\mathbb{R})$. *Then the function* $f *_{k,n} g$ *defined almost everywhere on* $\mathbb{R}$ *by*

$$f *_{k,n} g(x) = \int_{\mathbb{R}} \tau_x^{k,n} f((-1)^n y) g(y) d\gamma_{k,n}(y)$$

belongs to $L^2_{k,n}(\mathbb{R})$.

(ii) *Assume that* $1 \leqslant p, q, r \leqslant \infty$ *satisfy* $\frac{1}{p} + \frac{1}{q} - 1 = \frac{1}{r}$. *Then, for every* f *in* $L^p_{k,n}(\mathbb{R})$ *and* $g \in L^q_{k,n}(\mathbb{R})$, *the convolution product* $f *_{k,n} g$ *belongs to* $L^r_{k,n}(\mathbb{R})$ *and*

$$\|f *_{k,n} g\|_{L^r_{k,n}(\mathbb{R})} \leqslant 4 \|f\|_{L^p_{k,n}(\mathbb{R})} \|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{3.22}$$

(iii) *For* $f \in L^2_{k,n}(\mathbb{R})$ *and* $g \in L^1_{k,n}(\mathbb{R})$, *we have*

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f) \mathcal{F}_{k,n}(g). \tag{3.23}$$

Proof. of Theorem 3.3. Let f be a bounded and positive function in $L^1_{k,n,e}(\mathbb{R})$. In particular $f \in L^2_{k,n}(\mathbb{R})$, Therefore, we may consider the function $f *_{k,n} \alpha_t$, $t > 0$. Using Proposition 3.3 we prove that the previous function belongs to $L^1_{k,n}(\mathbb{R})$. On the other hand involving (3.23), Cauchy-Schwarz's inequality and Plancherel's formula (2.9), we infer that $\mathcal{F}_{k,n}(f *_{k,n} \alpha_t)$ belongs to $L^1_{k,n}(\mathbb{R})$. Thus $f *_{k,n} \alpha_t$ belongs to $\mathcal{W}_{k,n}(\mathbb{R})$. As the function f is an even positive function we deduce also that $f *_{k,n} \alpha_t$ is an even positive function. The positivity of the generalized translation operator on the positive even function of $\mathcal{W}_{k,n}(\mathbb{R})$ implies that

$$\forall\, t > 0, \quad \tau_x^{k,n}\big(f *_{k,n} \alpha_t\big) \geq 0. \tag{3.24}$$

Involving Plancherel's formula (2.9), the formula (2.8) and by simple calculation we see that

$$\lim_{t \to 0} ||f - f *_{k,n} \alpha_t||_{L^2_{k,n}(\mathbb{R})} = ||\mathcal{F}_{k,n}(f)\big(e^{-nt|\xi|^{\frac{2}{n}}} - 1\big)||_{L^2_{k,n}(\mathbb{R})} = 0.$$

Using similar ideas as above and (3.2), we prove that

$$\lim_{t \to 0} ||\tau_x^{k,n}\big(f - f *_{k,n} \alpha_t\big)||_{L^2_{k,n}(\mathbb{R})} = 0. \tag{3.25}$$

Thus up to sequences, (3.24) and (3.25) give that

$$\tau_x^{k,n} f(y) = \lim_{t \to 0} \tau_x^{k,n}\big(f *_{k,n} \alpha_t\big)(y) \geq 0$$

almost everywhere $y \in \mathbb{R}$. This finishes the proof of the first part of statement (i). For the second part of (i), applying the monotone convergence theorem to the relation (3.5) with $g(y) = e^{-\lambda|y|^{\frac{1}{n}}}$ and using by the same argument used in Proposition 3.3 we prove that

$$\begin{aligned}\int_{\mathbb{R}} \tau_x^{k,n} f(y) d\gamma_{k,n}(y) &= \lim_{\lambda\to 0}\int_{\mathbb{R}} \tau_x^{k,n} f(y) e^{-\lambda|y|^{\frac{1}{n}}} d\gamma_{k,n}(y)\\ &= \lim_{\lambda\to 0}\int_{\mathbb{R}} f(y)\tau_{(-1)^n x}^{k,n}\big(e^{-\lambda|y|^{\frac{1}{n}}}\big) d\gamma_{k,n}(y)\\ &= \int_{\mathbb{R}} f(y) d\gamma_{k,n}(y).\end{aligned}$$

(ii) If $f \in L^1_{k,n,e}(\mathbb{R}) \cap L^\infty_{k,n}(\mathbb{R})$, the previous result implies that

$$||\tau_x^{k,n} f||_{L^1_{k,n}(\mathbb{R})} \leqslant ||\tau_x^{k,n}|f|\,||_{L^1_{k,n}(\mathbb{R})} = ||f||_{L^1_{k,n}(\mathbb{R})}.$$

On the other hand if $f \in L^2_{k,n}(\mathbb{R})$, from (3.2) we have

$$||\tau_x^{k,n} f||_{L^2_{k,n}(\mathbb{R})} \leqslant ||f||_{L^2_{k,n}(\mathbb{R})}.$$

Thus by interpolation we deduce that for any $p \in [1,2]$

$$||\tau_x^{k,n} f||_{L^p_{k,n}(\mathbb{R})} \leqslant ||f||_{L^p_{k,n}(\mathbb{R})}.$$

Finally, by duality we infer the result.

(iii) Choose even functions $f_j \in \mathcal{W}_{k,n}(\mathbb{R})$ such that $f_j \to f$ and $\tau_x^{k,n} f_j \to \tau_x^{k,n} f$ in $L^1_{k,n}(\mathbb{R})$. Since

$$\int_{\mathbb{R}} \tau_x^{k,n} f_j(y) g(y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f_j(y)\tau_{(-1)^n x}^{k,n} g(y) d\gamma_{k,n}(y)$$

for every $g \in \mathcal{W}_{k,n}(\mathbb{R})$ we get, taking limit as j tends to infinity,

$$\int_{\mathbb{R}} \tau_x^{k,n} f(y) g(y) d\gamma_{k,n}(y) = \int_{\mathbb{R}} f(y)\tau_{(-1)^n x}^{k,n} g(y) d\gamma_{k,n}(y).$$

Now take $g(y) = e^{-\lambda|y|^{\frac{1}{n}}}$ and using the same argument used in Proposition 3.3 we prove the result. □

Involving Theorem 3.3 we improve the estimate given in Proposition 3.4 ii). More precisely, we have:

Corollary 3.2. *Assume that* $1 \leqslant p,q,r \leqslant \infty$ *satisfy* $\frac{1}{p}+\frac{1}{q}-1=\frac{1}{r}$. *Then, for every* $f \in L^p_{k,n,e}(\mathbb{R})$ *and* $g \in L^q_{k,n}(\mathbb{R})$, *the convolution product* $f *_{k,n} g$ *belongs to* $L^r_{k,n}(\mathbb{R})$ *and*

$$\|f *_{k,n} g\|_{L^r_{k,n}(\mathbb{R})} \leqslant \|f\|_{L^p_{k,n}(\mathbb{R})}\|g\|_{L^q_{k,n}(\mathbb{R})}. \tag{3.26}$$

We close this section by recalling the following results which will play a significant role.

Proposition 3.5. *([24]) (i) Let* f *and* g *in* $L^2_{k,n}(\mathbb{R})$. *Then* $f *_{k,n} g \in L^2_{k,n}(\mathbb{R})$ *if and only if* $\mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g)$ *belongs to* $L^2_{k,n}(\mathbb{R})$, *and in this case we have*

$$\mathcal{F}_{k,n}(f *_{k,n} g) = \mathcal{F}_{k,n}(f)\mathcal{F}_{k,n}(g).$$

(ii) Let f and g be in $L^2_{k,n}(\mathbb{R})$. Then, we have

$$\int_{\mathbb{R}} |f *_{k,n} g(x)|^2 d\gamma_{k,n}(x) = \int_{\mathbb{R}} |\mathcal{F}_{k,n}(f)(\xi)|^2 |\mathcal{F}_{k,n}(g)(\xi)|^2 d\gamma_{k,n}(\xi) \tag{3.27}$$

whenever both sides are finite.

4. DEFORMED STOCKWELL TRANSFORMS

Definition 4.1. *For any function h in $L^2_{k,n,e}(\mathbb{R})$ and any $\nu \in \mathbb{R}$, we define the modulation of h by ν as:*

$$\mathcal{M}_\nu h := \mathcal{F}_{k,n}(\sqrt{\tau_\nu^{k,n}(|\mathcal{F}_{k,n}(h)|^2)}), \tag{4.1}$$

where $\tau_\nu^{k,n}$, $\nu \in \mathbb{R}$, are the generalized translation operators.

Let $a \in \mathbb{R}$. The dilation operator Δ_a of a measurable function h, is defined by

$$\forall\, x \in \mathbb{R},\ \Delta_a h(x) := |a|^{\frac{(2k-1)n+2}{2n}} h(ax). \tag{4.2}$$

By simple calculations we prove that these operators satisfy the following properties.

Proposition 4.1. *(i) For all a, b in $\mathbb{R}^*$, we have*

$$\Delta_a \Delta_b = \Delta_{ab} \tag{4.3}$$

and

$$\Delta_a M_b = M_{ab} \Delta_a. \tag{4.4}$$

(ii) Let $a \in \mathbb{R}^*$. For all h in $L^2_{k,n}(\mathbb{R})$, the function $\Delta_a h$ belongs to $L^2_{k,n}(\mathbb{R})$ and we have

$$||\Delta_a h||_{L^2_{k,n}(\mathbb{R})} = ||h||_{L^2_{k,n}(\mathbb{R})} \tag{4.5}$$

and

$$\mathcal{F}_{k,n}(\Delta_a h)(y) = |a|^{-\frac{(2k-1)n+2}{2n}} \mathcal{F}_{k,n}(h)(\frac{y}{a}), \quad y \in \mathbb{R}. \tag{4.6}$$

(iii) Let $a \in \mathbb{R}^*$. For all h, g in $L^2_{k,n}(\mathbb{R})$, we have

$$\langle \Delta_a h, g\rangle_{L^2_{k,n}(\mathbb{R})} = \langle h, \Delta_{\frac{1}{a}} g\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{4.7}$$

(iv) Let $a \in \mathbb{R}^*$ and $x \in \mathbb{R}$. We have

$$\Delta_a \tau_x^{k,n} = \tau_{\frac{x}{a}}^{k,n} \Delta_a. \tag{4.8}$$

(v) Let $a \in \mathbb{R}^*$ and $h \in L^2_{k,n}(\mathbb{R})$. We have

$$|\Delta_a h|^2 = |a|^{\frac{(2k-1)n+2}{2n}} \Delta_a |h|^2. \tag{4.9}$$

Definition 4.2. *A deformed Stockwell wavelet on $\mathbb{R}$ is an even measurable function h on $\mathbb{R}$ satisfying for almost all $\xi \in \mathbb{R}^*$, the condition*

$$0 < C_h := \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 d\gamma_{k,n}(\nu) < \infty. \tag{4.10}$$

Proposition 4.2. *Let h be a deformed Stockwell wavelet on $\mathbb{R}$, we have*

$$C_h := \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h(\xi)|^2 d\gamma_{k,n}(\nu) = M_{k,n} \int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big)(\frac{(-1)^n\xi}{\nu})\frac{d\nu}{|\nu|}.$$

Proof. Let $\nu \in \mathbb{R}^*$. Using the relations (4.1), (4.2), (4.6), (4.8) and (4.9) we deduce that

$$\begin{array}{rcl} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 &=& \tau_\nu^{k,n}(|\mathcal{F}_{k,n}(\Delta_\nu h)|^2)((-1)^n\xi) \\ &=& \frac{1}{|\nu|^{\frac{(2k-1)n+2}{n}}}\tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big)(\frac{(-1)^n\xi}{\nu}). \end{array} \tag{4.11}$$

Then (4.10) is written as

$$\begin{array}{rcl} C_h : &=& \int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu \Delta_\nu h)(\xi)|^2 d\gamma_{k,n}(\nu) \\ &=& \int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big)(\frac{(-1)^n\xi}{\nu})\frac{d\gamma_{k,n}(\nu)}{|\nu|^{\frac{(2k-1)n+2}{n}}} \\ &=& M_{k,n}\int_{\mathbb{R}} \tau_1^{k,n}\big(|\mathcal{F}_{k,n}(h)|^2\big)(\frac{(-1)^n\xi}{\nu})\frac{d\nu}{|\nu|}. \end{array}$$

Thus we obtain the desired result. □

Let $\nu \in \mathbb{R}^*$ and h be a deformed Stockwell wavelet in $L^2_{k,n}(\mathbb{R})$. We consider the family $h_{x,\nu}$, $x \in \mathbb{R}$, of functions on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$ defined by

$$h_{x,\nu}(y) := \tau_x^{k,n}\mathcal{M}_\nu(\Delta_\nu h)((-1)^n y),\ y \in \mathbb{R}, \tag{4.12}$$

where $\tau_x^{k,n}$, $x \in \mathbb{R}$, are the generalized translation operators given by (3.1).

We note that we have

$$\forall\,(x,\nu) \in \mathbb{R}^2, \quad ||h_{x,\nu}||_{L^2_{k,n}(\mathbb{R})} \leqslant ||h||_{L^2_{k,n}(\mathbb{R})}. \tag{4.13}$$

For $1 \leqslant p \leqslant \infty$, let $L^p_{\mu_{k,n}}(\mathbb{R}^2)$, $p \in [1,\infty]$, be the space of measurable functions f on $\mathbb{R}^2$ such that

$$\begin{array}{rcl} \|f\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} &:=& \left(\int_{\mathbb{R}^2} |f(x,\nu)|^p d\mu_k(x,\nu)\right)^{\frac{1}{p}} < \infty, \quad 1 \leqslant p < \infty, \\ \|f\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} &:=& \operatorname*{ess\,sup}_{(x,\nu)\in\mathbb{R}^2} |f(x,\nu)| < \infty, \end{array}$$

where the measure $\mu_{k,n}$ is defined by

$$\forall\,(x,\nu) \in \mathbb{R}^2, \quad d\mu_{k,n}(x,\nu) = d\gamma_{k,n}(x)d\gamma_{k,n}(\nu).$$

Definition 4.3. *Let h be a deformed Stockwell wavelet on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$. The deformed Stockwell continuous transform $\mathcal{S}_h^{k,n}$ on $\mathbb{R}$ is defined for regular functions f on $\mathbb{R}$ by*

$$\forall\,(x,\nu) \in \mathbb{R}^2, \quad \mathcal{S}_h^{k,n}(f)(x,\nu) = \int_{\mathbb{R}} f(y)\overline{h_{x,\nu}(y)}d\gamma_{k,n}(y). \tag{4.14}$$

This transform can also be written in the form

$$\mathcal{S}_h^{k,n}(f)(x,\nu) = f *_{k,n} \overline{\mathcal{M}_\nu \Delta_\nu h}(x), \tag{4.15}$$

where $*_{k,n}$ is the generalized convolution product given by (3.21).

Remark 4.1. *(i) Let h be a deformed Stockwell wavelet in $L^2_{k,n}(\mathbb{R})$. Using relation (4.14), Cauchy-Schwarz's inequality and relation (4.13) we get, for all f in $L^2_{k,n}(\mathbb{R})$*

$$\|\mathcal{S}_h^{k,n}(f)\|_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{4.16}$$

(ii) Using Proposition 4.1 and by a standard computation it is easy to see that, for every $f \in L^2_{k,n}(\mathbb{R})$ and h in $L^2_{k,n,e}(\mathbb{R})$, for all $\lambda > 0$ and for all $(x,\nu) \in \mathbb{R}^2$, we have

$$\mathcal{S}_h^{k,n}(f_\lambda)(x,\nu) = \mathcal{S}_h^{k,n}(f)(\frac{x}{\lambda}, \lambda\nu), \tag{4.17}$$

where

$$\forall\, t > 0,\ \forall\, x \in \mathbb{R},\ g_t(x) := \frac{1}{t^{\frac{(2k-1)n+2}{2n}}} g(\frac{x}{t}).$$

Henceforth, the function h will denote a deformed Stockwell wavelet on $\mathbb{R}$ in $L^2_{k,n}(\mathbb{R})$. By simple calculations we prove the following:

Lemma 4.1. *For any $f \in L^2_{k,n}(\mathbb{R})$, we have*

$$\mathcal{F}_{k,n}\Big(\mathcal{S}_h^{k,n}(f)(.,\nu)\Big)(\xi) = \mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)\mathcal{F}_{k,n}(f)(\xi). \tag{4.18}$$

Theorem 4.1. *(Parseval's formula for $\mathcal{S}_h^{k,n}$). Let f, g in $L^2_{k,n}(\mathbb{R})$. Then, we have*

$$\int_{\mathbb{R}} f(x)\overline{g(x)}d\gamma_{k,n}(x) = \frac{1}{C_h}\int_{\mathbb{R}^2} \mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)}d\mu_{k,n}(x,\nu). \tag{4.19}$$

Proof. Using Fubini's Theorem, relation (4.15) and Parseval's formula (2.10), we get

$$\int_{\mathbb{R}^2} \mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)}d\mu_{k,n}(x,\nu) =$$

$$\int_{\mathbb{R}^2} \Big(f *_{k,n} \overline{\mathcal{M}_\nu\Delta_\nu h}(x)\Big)\Big(\overline{g *_{k,n} \overline{\mathcal{M}_\nu\Delta_\nu h}(x)}\Big)d\mu_{k,n}(x,\nu) =$$

$$\int_{\mathbb{R}}\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}|\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)|^2 d\gamma_{k,n}(\xi)d\gamma_{k,n}(\nu) =$$

$$\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}\Big(\int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu\Delta_\nu h)((-1)^n\xi)|^2 d\gamma_{k,n}(\nu)\Big)d\gamma_{k,n}(\xi).$$

As h is a deformed Stockwell wavelet, (4.10) give that

$$\int_{\mathbb{R}} |\mathcal{F}_{k,n}(\mathcal{M}_\nu\Delta_\nu h)((-1)^n\xi)|^2 d\gamma_{k,n}(\nu) \quad = \quad C_h.$$

Thus we obtain

$$\int_{\mathbb{R}^2} \mathcal{S}_h^{k,n}(f)(x,\nu)\overline{\mathcal{S}_h^{k,n}(g)(x,\nu)}d\mu_{k,n}(x,\nu) = C_h\int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}d\gamma_{k,n}(\xi).$$

Finally using Parseval's formula (2.10) we obtain the result. □

Corollary 4.1. *(Plancherel's formula for $\mathcal{S}_h^{k,n}$). For all f in $L^2_{k,n}(\mathbb{R})$ we have*

$$\int_{\mathbb{R}} |f(x)|^2 d\gamma_{k,n}(x) = \frac{1}{C_h}\int_{\mathbb{R}^2} |\mathcal{S}_h^{k,n}(f)(x,\nu)|^2 d\mu_{k,n}(x,\nu). \tag{4.20}$$

By Riesz-Thorin's interpolation theorem we derive the following.

Proposition 4.3. *Let $f \in L^2_{k,n}(\mathbb{R})$ and p belongs in $[2,\infty]$. We have*

$$\|\mathcal{S}_h^{k,n}(f)\|_{L^p_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant (C_h)^{\frac{1}{p}} (\|h\|_{L^2_{k,n}(\mathbb{R})})^{\frac{p-2}{p}} \|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{4.21}$$

5. PRACTICAL REAL INVERSION FORMULAS FOR $\mathcal{S}_h^{k,n}$

5.1. Tikhonov regularization. Nowadays, the general theory of reproducing kernels have found many applications to Integral transforms, Inverse problems, Integral equations, Inversions for a family of bounded linear operators, Sampling theory, Linear differential equations with variable coefficients, Approximations of functions. Arguing from these point of view, many works were done on them, we refer in particular to the papers of Saitoh et al. [5, 32, 33].

Before the applications to the Tiknohov regularization, we shall examine the concept of the Moore-Penrose generalized inverses from the viewpoint of the theory of reproducing kernels. Here, we will be able to realize the natural and powerful method of the theory of reproducing kernels for the best approximation problems that lead to the Moore-Penrose generalized inverses.

Let E be an arbitrary set and let H_K be a reproducing kernel Hilbert space admitting the reproducing kernel K on E. For any Hilbert space $\mathcal{H}$ we consider a bounded linear operator L from H_K to $\mathcal{H}$. Then the following problem is a classical and fundamental problem which is known as best approximate mean square norm problems: For any member d of $\mathcal{H}$

$$\inf_{f\in H_K} ||Lf - d||_{\mathcal{H}}. \tag{5.1}$$

This problem carries, however, a complicated structure, when the Hilbert spaces are infinite dimensions and the problem leads to the generalized inverse in the sense of Moore-Penrose. However, this extremal problem is involved in both the existence of the extremal functions in (5.1) and their representations. So, we shall consider its Tikhonov regularization. We start it with the following fundamental theorem.

Theorem 5.1. *([32].) Let H_K be a Hilbert space admitting the reproducing kernel $K(p,q)$ on a set E and $\mathcal{H}$ an Hilbert space. Let $L : H_K \to \mathcal{H}$ be a bounded linear operator. For $r > 0$, we introduce the inner product in H_K and we call it H_{K_r} as*

$$\langle f_1, f_2\rangle_{H_{K_r}} = r\langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}}.$$

Then:

i*) H_{K_r} is a Hilbert space with the reproducing kernel $K_r(p,q)$ on E and satisfying the equation*

$$K_r(.,q) = (rI + L^*L)K(.,q),$$

where L^* is the adjoint operator of L.

ii*) For any* $r > 0$ *and for any* h *in* $\mathcal{H}$*, the infimum*

$$\inf_{f\in H_K}\left\{r\|f\|^2_{H_K} + \|Lf - h\|^2_{\mathcal{H}}\right\}$$

is attained by a unique function $f^*_{r,h}$ *in* H_K *and this extremal function is given by*

$$f^*_{r,h}(p) = \langle h, LK_r(.,p)\rangle_{\mathcal{H}}. \tag{5.2}$$

In this Section by applying the general theory of reproducing kernels and in particular the Tikhonov regularization, we shall consider the practical constructions of approximate solutions for bounded linear operator equations involving the deformed Stockwell transform. The functional spaces used in our analyse are the generalized Sobolev spaces that are built from the deformed Hankel transform and deformed Stockwell transform and that are the typical Hilbert spaces in our setting.

5.2. Reproducing kernels.
Notation. Let us denote by
$\mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}$.
$\mathcal{S}'(\mathbb{R})$ the topological dual of the Schwartz space $\mathcal{S}(\mathbb{R})$.

Remark 5.1. *We note that Johansen in [[17], Lemma 2.12] has proved that the Schwartz space is invariant under the deformed Hankel transform.*

Definition 5.1. *The deformed Hankel transform of a distribution* τ *in* $\mathcal{S}'(\mathbb{R})$ *is defined by*

$$\langle \mathcal{F}_{k,n}(\tau), \phi\rangle = \langle \tau, \mathcal{F}^{-1}_{k,n}(\phi)\rangle, \quad \text{for all } \phi \in \mathcal{S}(\mathbb{R}). \tag{5.3}$$

Definition 5.2. *Let* $s \in \mathbb{R}$*, we define the generalized Sobolev space* $W^s_{k,n}(\mathbb{R})$ *as*

$$\left\{u \in \mathcal{S}'(\mathbb{R}) : (1 + |\xi|^2)^{\frac{s}{2}}\mathcal{F}_{k,n}(u) \in L^2_{k,n}(\mathbb{R})\right\}.$$

We provided this space with inner product $\langle ., .\rangle_{W^s_{k,n}(\mathbb{R})}$ given by:

$$\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}}(1 + |\xi|^2)^s\mathcal{F}_{k,n}(f)(\xi)\overline{\mathcal{F}_{k,n}(g)(\xi)}d\gamma_{k,n}(\xi), \quad \text{for all } f, g \in W^s_{k,n}(\mathbb{R}). \tag{5.4}$$

The norm associated to the inner product is defined by:

$$\|f\|_{W^s_{k,n}(\mathbb{R})} := \Big(\int_{\mathbb{R}}(1 + |\xi|^2)^s|\mathcal{F}_{k,n}(f)(\xi)|^2d\gamma_{k,n}(\xi)\Big)^{\frac{1}{2}}.$$

Proposition 5.1. *For* $s > \frac{(2k-1)n+2}{2n}$*, the generalized Sobolev space* $W^s_{k,n}(\mathbb{R})$ *admits the following reproducing kernel:*

$$K_s(x, y) = \int_{\mathbb{R}}\frac{B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi, y)}{(1 + |\xi|^2)^s}d\gamma_{k,n}(\xi),$$

(i) for all $y \in \mathbb{R}$*, the function* $x \mapsto K_s(x, y)$ *belongs to* $W^s_{k,n}(\mathbb{R})$*,*
(ii) the reproducing property: for all $f \in W^s_{k,n}(\mathbb{R})$ *and* $y \in \mathbb{R}$*,*

$$f(y) = \langle f, K_s(x, y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

Proof. i) Let y be in $\mathbb{R}$. It is easy to see that the function

$$\Upsilon_y : \xi \mapsto \frac{B_{k,n}(\xi, y)}{(1+|\xi|^2)^s}$$

belongs to $L^1_{k,n}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$ when $s > \frac{(2k-1)n+2}{2n}$. Thus the function K_s is well defined and we can write

$$K_s(x, y) = \mathcal{F}^{-1}_{k,n}(\Upsilon_y)(x), \quad \text{for all } x \in \mathbb{R}.$$

Moreover, from Proposition 2.2, we can see that the function $K_s(., y)$ belongs to $L^2_{k,n}(\mathbb{R})$, and we have

$$\mathcal{F}_{k,n}\big(K_s(., y)\big)(\xi) = \frac{B_{k,n}(\xi, y)}{(1+|\xi|^2)^s}. \tag{5.5}$$

As $B_{k,n}(\xi, y)$ is bounded, we obtain

$$|\mathcal{F}_{k,n}\big(K_s(., y)\big)(\xi)| \leqslant \frac{1}{(1+|\xi|^2)^s}$$

and

$$\begin{aligned} ||K_s(., y)||_{W^s_{k,n}(\mathbb{R})} &\leqslant C(k, n, s) := \Big(\int_{\mathbb{R}} \frac{d\gamma_{k,n}(\xi)}{(1+|\xi|^2)^s}\Big)^{\frac{1}{2}} \\ &= \Big(\frac{\Gamma(\frac{(2k-1)n+2}{2n})\Gamma(s-\frac{(2k-1)n+2}{2n})}{\Gamma(s)}\Big)^{\frac{1}{2}} < \infty. \end{aligned} \tag{5.6}$$

This proves that for all $y \in \mathbb{R}$ the function $K_s(., y)$ belongs to $W^s_{k,n}(\mathbb{R})$.

(ii) Let f be in $W^s_{k,n}(\mathbb{R})$ and y in $\mathbb{R}$. From (5.4) and (5.5), we have

$$\langle f, K_s(., y)\rangle_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}} \mathcal{F}_{k,n}(f)(\xi) B_{k,n}(y, (-1)^n\xi) d\gamma_{k,n}(\xi), \tag{5.7}$$

and from inversion formula, we obtain the reproducing property

$$f(y) = \langle f, K_s(x, y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

This completes the proof of the proposition. □

Corollary 5.1. *For $s > \frac{(2k-1)n+2}{2n}$, the generalized Sobolev space $W^s_{k,n}(\mathbb{R})$ is embedded in $C(\mathbb{R})$.*

5.3. Extremal functions associated with the partial deformed Stockwell transform.

Notation. Let h be in $L^2_{k,n,e}(\mathbb{R})$ and let $\nu \in \mathbb{R}^*$. We denote by $\mathcal{P}^{k,n}_{h,\nu}$ the partial deformed Stockwell transform defined by

$$\mathcal{P}^{k,n}_{h,\nu}(f) := \mathcal{S}^{k,n}_h(f)(., \nu), \quad \text{for all } f \in L^2_{k,n}(\mathbb{R}).$$

Proposition 5.2. *Let h be in $L^1_{k,n,e}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$ and let $\nu \in \mathbb{R}^*$. The transformation $\mathcal{P}^{k,n}_{h,\nu}$ is a bounded linear operator from $W^s_{k,n}(\mathbb{R})$, $s \geq 0$, into $L^2_{k,n}(\mathbb{R})$, and there exist a positive constant $C_{k,n}(\nu, h)$ such that we have*

$$\|\mathcal{P}^{k,n}_{h,\nu}(f)\|_{L^2_{k,n}(\mathbb{R})} \leqslant C_{k,n}(\nu, h)\|f\|_{W^s_{k,n}(\mathbb{R})}, \quad f \in W^s_{k,n}(\mathbb{R}).$$

Proof. Using the relations (4.18), (4.11), (3.19) and Proposition 3.5 ii) we obtain the result. □

Let $r>0$, $s\geq 0$, $\nu\in\mathbb{R}^*$ and h be in $L^1_{k,n,e}(\mathbb{R})\cap L^2_{k,n}(\mathbb{R})$. We introduce the inner product in the space $W^s_{k,n}(\mathbb{R})$

$$\langle f,g\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}=r\langle f,g\rangle_{W^s_{k,n}(\mathbb{R})}+\langle\mathcal{P}^{k,n}_{h,\nu}(f),\mathcal{P}^{k,n}_{h,\nu}(g)\rangle_{L^2_{k,n}(\mathbb{R})},\quad f,g\in W^s_{k,n}(\mathbb{R}).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}:=r\|f\|^2_{W^s_{k,n}(\mathbb{R})}+\|\mathcal{P}^{k,n}_{h,\nu}(f)\|^2_{L^2_{k,n}(\mathbb{R})}.$$

Remark 5.2. *Simple calculations give that* $||.||_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}$ *and* $||.||_{W^s_{k,n}(\mathbb{R})}$ *are equivalent for* $r>0$ *and* $\nu\in\mathbb{R}$.

Proposition 5.3. *Let* $r>0$, $\nu\in\mathbb{R}^*$, $s>\frac{(2k-1)n+2}{2n}$ *and* h *be a deformed Stockwell wavelet in* $L^1_{k,n,e}(\mathbb{R})\cap L^2_{k,n}(\mathbb{R})$. *Then the generalized Sobolev space* $(W^s_{k,n}(\mathbb{R}),\langle.,.\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})})$, *possesses a reproducing kernel* $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}$ *satisfying the identity*

$$\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)=\Big(rI+(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu}\Big)^{-1}K_s(.,y)\tag{5.8}$$

where $(\mathcal{P}^{k,n}_{h,\nu})^*:L^2_{k,n}(\mathbb{R})\longrightarrow W^s_{k,n}(\mathbb{R})$ *is the adjoint operator of* $\mathcal{P}^{k,n}_{h,\nu}$ *given by*

$$\langle\mathcal{P}^{k,n}_{h,\nu}(f),g\rangle_{L^2_{k,n}(\mathbb{R})}=\langle f,(\mathcal{P}^{k,n}_{h,\nu})^*g\rangle_{W^s_{k,n}(\mathbb{R})},\quad f\in W^s_{k,n}(\mathbb{R}),\ g\in L^2_{k,n}(\mathbb{R}).$$

Moreover the kernel $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}$ *satisfies the following properties*

(i) $||\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)||_{W^s_{k,n}(\mathbb{R})}\leqslant\frac{C(k,n,s)}{r}$, *for all* $y\in\mathbb{R}$.

(ii) $||\mathcal{P}^{k,n}_{h,\nu}\big(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\big)||_{L^2_{k,n}(\mathbb{R})}\leqslant\frac{C(k,n,s)}{\sqrt{2r}}$, *for all* $y\in\mathbb{R}$.

(iii) $||(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))||_{W^s_{k,n}(\mathbb{R})}\leqslant C(k,n,s)$, *for all* $y\in\mathbb{R}$, *where* $C(k,n,s)$ *is the constant given by (5.6).*

Proof. From Corollary 5.1, Proposition 5.2 and Remark 5.2, we deduce that the map $u\mapsto u(y)$, $y\in\mathbb{R}$, is a continuous linear functional on the space $W^s_{k,n}(\mathbb{R})$ equipped with the inner product $\langle.,.\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}$.
Thus from [32], $(W^s_{k,n}(\mathbb{R}),\langle.,.\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})})$ has a reproducing kernel denoted by $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}$. On the other hand, we have

$$\begin{aligned}f(y)&=\langle f,\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})}\\&=r\langle f,\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}+\langle\mathcal{P}^{k,n}_{h,\nu}(f),\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{L^2_{k,n}(\mathbb{R})}\\&=\langle f,(rI+(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu})\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}.\end{aligned}$$

Thus,

$$(rI+(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu})\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)=K_s(.,y).\tag{5.9}$$

Furthermore, the previous identity implies that

$$\begin{aligned}&r^2||\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)||^2_{W^s_{k,n}(\mathbb{R})}+2r||\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))||^2_{L^2_{k,n}(\mathbb{R})}\\&+||(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))||^2_{W^s_{k,n}(\mathbb{R})}\\&=||K_s(.,y)||^2_{W^s_{k,n}(\mathbb{R})}.\end{aligned}$$

From this relation and using the fact that

$$||K_s(.,y)||_{W^s_{k,n}(\mathbb{R})} \leqslant C(k,n,s),$$

we obtain the properties (i), (ii) and (iii). □

Remark 5.3. *Using similar ideas as in Proposition 5.1, we prove that*

$$\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(x,y) = \int_{\mathbb{R}} \frac{B_{k,n}((-1)^n\xi,x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)|^2} d\gamma_{k,n}(\xi).$$

We can now state the main result of this paragraph.

Theorem 5.2. *Let $s > \frac{(2k-1)n+2}{2n}$ and h be in $L^1_{k,n,e}(\mathbb{R}) \cap L^2_{k,n}(\mathbb{R})$.*

(i) For any $g \in L^2_{k,n}(\mathbb{R})$ and for any $r > 0$, $\nu \in \mathbb{R}^*$ the best approximate function $f^*_{r,\nu,g}$ in the sense

$$\inf_{f\in W^s_{k,n}(\mathbb{R})} \left\{ r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|g - \mathcal{P}^{k,n}_{h,\nu}(f)\|^2_{L^2_{k,n}(\mathbb{R})} \right\} \tag{5.10}$$

exists uniquely and it is represented by

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,\nu,g}(y) = \langle g, \mathcal{P}^{k,n}_{h,\nu}\big(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\big)\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{5.11}$$

(ii) The extremal function $f^*_{r,\nu,g}$ satisfies the following inequality:

$$\forall\, y \in \mathbb{R}, \quad |f^*_{r,\nu,g}(y)| \leqslant \frac{C(k,n,s)}{\sqrt{2r}} ||g||_{L^2_{k,n}(\mathbb{R})}.$$

Proof. (i) The existence and uniqueness of extremal function $f^*_{r,\nu,g}$ satisfying (5.10) is given by [33], and the extremal function $f^*_{r,\nu,g}$ is represented by

$$f^*_{r,\nu,g}(y) = \langle g, \mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{L^2_{k,n}(\mathbb{R})}, \quad y \in \mathbb{R}.$$

(ii) From Proposition 5.3 (ii), we have

$$|f^*_{r,\nu,g}(y)| \leqslant ||g||_{L^2_{k,n}(\mathbb{R})} ||\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))||_{L^2_{k,n}(\mathbb{R})} \leqslant \frac{C(k,n,s)}{\sqrt{2r}} ||g||_{L^2_{k,n}(\mathbb{R})}.$$

Thus the theorem is proved. □

Corollary 5.2. *Let $s > \frac{(2k-1)n+2}{2n}$, $r > 0$ and $\nu \in \mathbb{R}^*$. If f is in $W^s_{k,n}(\mathbb{R})$ and $g = \mathcal{P}^{k,n}_{h,\nu}(f)$. Then*

(i) $f(y) = \lim\limits_{r\to 0^+} f^*_{r,\nu,g}(y)$, for all $y \in \mathbb{R}$.

(ii) $|f(y) - f^*_{r,\nu,g}(y)| \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}$, for all $y \in \mathbb{R}$.

(iii) $|f^*_{r,\nu,g}(y)| \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}$, for all $y \in \mathbb{R}$.

Proof. Let f be in $W^s_{k,n}(\mathbb{R})$.

(i) Then

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,\nu,g}(y) = \langle f, (\mathcal{P}^{k,n}_{h,\nu})^* \mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))\rangle_{W^s_{k,n}(\mathbb{R})}. \tag{5.12}$$

But from (5.9), we have

$$\forall\, y \in \mathbb{R}, \quad \lim_{r\to 0^+} (\mathcal{P}^{k,n}_{h,\nu})^* \mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)) = K_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f^*_{r,\nu,g}(y) = \langle f, K_s(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} = f(y).$$

(ii) From (5.9) and (5.12), the extremal function $f^*_{r,\nu,g}$ satisfies

$$\forall\, y\in\mathbb{R},\quad f^*_{r,\nu,g}(y) = f(y) - r\langle f, \mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

Thus by Proposition 5.3 (i), we obtain for all $y\in\mathbb{R}$

$$|f^*_{r,\nu,g}(y) - f(y)| \leqslant r||f||_{W^s_{k,n}(\mathbb{R})}||\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)||_{W^s_{k,n}(\mathbb{R})} \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}.$$

(iii) From (5.12) and Proposition 5.3 (iii), the extremal function $f^*_{r,\nu,g}$ satisfies: for all $y\in\mathbb{R}$

$$|f^*_{r,\nu,g}(y)| \leqslant ||f||_{W^s_{k,n}(\mathbb{R})}||(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu}(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))||_{W^s_{k,n}(\mathbb{R})} \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}.$$

□

Remark 5.4. *Let $s > \frac{(2k-1)n+2}{2n}$, $r>0$ and $\nu\in\mathbb{R}^*$. If $\mathcal{P}^{k,n}_{h,\nu}$ is isometry (i.e. $(\mathcal{P}^{k,n}_{h,\nu})^*\mathcal{P}^{k,n}_{h,\nu} = Id$), then*

(i) $\langle .,.\rangle_{\mathcal{P}^{k,n}_{h,\nu},r,W^s_{k,n}(\mathbb{R})} = (r+1)\langle .,.\rangle_{W^s_{k,n}(\mathbb{R})}$.

(ii) $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(x,y) = \frac{1}{r+1}K_s(x,y)$, for all $x,y\in\mathbb{R}$.

(iii) For all $y\in\mathbb{R}$, $f^*_{r,\nu,g}(y) = \frac{1}{r+1}(\mathcal{P}^{k,n}_{h,\nu})^*g(y)$, $g\in L^2_{k,n}(\mathbb{R})$.

(iv) For all $y\in\mathbb{R}$, $f^*_{r,\nu,\mathcal{P}^{k,n}_{h,\nu}(u)}(y) = \frac{1}{r+1}u(y)$, $u\in W^s_{k,n}(\mathbb{R})$.

Proposition 5.4. *Let $s > \frac{(2k-1)n+2}{2n}$ and h be in $L^1_{k,n,e}(\mathbb{R})\cap L^2_{k,n}(\mathbb{R})$.*

i*) For any $g\in L^2_{k,n}(\mathbb{R})$ and for any $r>0$, $\nu\in\mathbb{R}^*$, the best approximate function $f^*_{r,\nu,g}$ is represented by*

$$f^*_{r,\nu,g}(y) = \int_{\mathbb{R}} g(x)Q_{r,\nu,h}(x,y)d\gamma_{k,n}(x),$$

where

$$Q_{r,\nu,h}(x,y) = \int_{\mathbb{R}} \frac{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)B_{k,n}((-1)^n\xi,x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)|^2}d\gamma_{k,n}(\xi). \tag{5.13}$$

ii*) If we take $g = \mathcal{P}^{k,n}_{h,\nu}(f)$, then*

$$\lim_{r\to 0^+} ||f^*_{r,\nu,g} - f||_{W^s_{k,n}(\mathbb{R})} = 0.$$

Moreover, $\{f^*_{r,\nu,g}\}_{r>0}$ converges uniformly to f as $r\to 0^+$.

iii*) Let $\delta>0$ and let g, g_δ satisfy $\|g-g_\delta\|_{L^2_{k,n}(\mathbb{R})} \leqslant \delta$. Then*

$$\|f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta}\|_{W^s_{k,n}(\mathbb{R})} \leqslant \frac{\delta}{2\sqrt{r}}.$$

Proof. i) By Remark 5.3 and Theorem 5.2 i), the infimum given by (5.11) is attained by a unique function $f^*_{r,\nu,g}$, and the extremal function $f^*_{r,\nu,g}$ is represented by

$$f^*_{r,\nu,g}(y) = \langle g, \mathcal{S}^{k,n}_h(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y))(.,\nu)\rangle_{L^2_{k,n}(\mathbb{R})},\quad y\in\mathbb{R},$$

where $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}$ is the kernel given by Remark 5.3.
On the other hand we have for all $x \in \mathbb{R}$,

$$\mathcal{S}^{k,n}_h(f)(x,\nu) = \int_{\mathbb{R}} \mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)\mathcal{F}_{k,n}(f)(\xi)B_{k,n}((-1)^n\xi, x)d\gamma_{k,n}(\xi).$$

Using the properties of the kernel $\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}$ and the definition of the deformed Stockwell transform, we get

$$\begin{aligned}\mathcal{S}^{k,n}_h\Big(\mathcal{K}_{\mathcal{P}^{k,n}_{h,\nu},r}(.,y)\Big)(x,\nu) &= \int_{\mathbb{R}} \frac{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)B_{k,n}((-1)^n\xi, x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2}d\gamma_{k,n}(\xi)\\ &= Q_{r,\nu,h}(x,y).\end{aligned}$$

This gives (5.13).

ii) From (5.13) and Fubini's theorem we have

$$\mathcal{F}_{k,n}(f^*_{r,\nu,g})(\xi) = \frac{\overline{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)}\mathcal{F}_{k,n}(g)(\xi)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2}.$$

Hence

$$\mathcal{F}_{k,n}(f^*_{r,\nu,g} - f)(\xi) = \frac{-r(1+|\xi|^2)^s\mathcal{F}_{k,n}(f)(\xi)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2}.$$

Then we obtain

$$\|f^*_{r,\nu,g} - f\|^2_{W^s_{k,n}(\mathbb{R})} = \int_{\mathbb{R}} h_{r,\nu,s}(\xi)|\mathcal{F}_{k,n}(f)(\xi)|^2 d\gamma_{k,n}(\xi),$$

with

$$h_{r,\nu,s}(\xi) = \frac{r^2(1+|\xi|^2)^{3s}}{\Big(r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2\Big)^2}.$$

Since

$$\lim_{r\to 0} h_{r,\nu,s}(\xi) = 0$$

and

$$|h_{r,\nu,s}(\xi)| \leqslant (1+|\xi|^2)^s,$$

we obtain the result from the dominated convergence theorem.

iii) From (5.13) and Fubini's theorem we have

$$\mathcal{F}_{k,n}(f^*_{r,\nu,g})(\xi) = \frac{\overline{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)}\mathcal{F}_{k,n}(g)(\xi)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2}. \tag{5.14}$$

Hence

$$\mathcal{F}_{k,n}(f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta})(\xi) = \frac{\overline{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)}\mathcal{F}_{k,n}(g-g_\delta)(\xi)}{r(1+|\xi|^2)^s + |\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi)|^2}.$$

Using the inequality $(x+y)^2 \geq 4xy$, we obtain

$$(1+|\xi|^2)^s\Big|\mathcal{F}_{k,n}(f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta})(\xi)\Big|^2 \leqslant \frac{1}{4r}|\mathcal{F}_{k,n}(g-g_\delta)(\xi)|^2.$$

Thus from Plancherel's formula (2.9) we obtain

$$\|f^*_{r,\nu,g} - f^*_{r,\nu,g_\delta}\|^2_{W^s_{k,n}(\mathbb{R})} \leqslant \frac{1}{4r}\|\mathcal{F}_{k,n}(g-g_\delta)\|^2_{L^2_{k,n}(\mathbb{R})} = \frac{1}{4r}\|g-g_\delta\|^2_{L^2_{k,n}(\mathbb{R})},$$

which gives the desired result. □

5.4. Extremal function associated with $\mathcal{S}_h^{k,n}$.
Let $r > 0$, $s \geq 0$ and h be in $L^2_{k,n,e}(\mathbb{R})$. We introduce the inner product in the space $W^s_{k,n}(\mathbb{R})$

$$\langle f, g\rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})} = r\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathcal{S}_h^{k,n}(f), \mathcal{S}_h^{k,n}(g)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \quad f, g \in W^s_{k,n}(\mathbb{R}).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})} := r\|f\|^2_{W^s_{k,n}(\mathbb{R})} + \|\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

From (4.19), the inner product $\langle ., , \rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})}$ can be written as

$$\langle f, g\rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})} = r\langle f, g\rangle_{W^s_{k,n}(\mathbb{R})} + C_h\langle f, g\rangle_{L^2_{k,n}(\mathbb{R})}. \tag{5.15}$$

Remark 5.5. *Simple calculations give that* $||.||_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})}$ *and* $||.||_{W^s_{k,n}(\mathbb{R})}$ *are equivalent for* $r > 0$.

Proposition 5.5. *Let* $s > \frac{(2k-1)n+2}{2n}$ *and* h *be in* $L^2_{k,n,e}(\mathbb{R})$*. Then the generalized Sobolev space* $(W^s_{k,n}(\mathbb{R}), \langle ., .\rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})})$*, possesses the reproducing kernel* $\mathfrak{K}^{k,n}_{r,h}$ *satisfying the following identity*

$$\mathfrak{K}^{k,n}_{r,h}(., y) = \Big(rI + (\mathcal{S}_h^{k,n})^*\mathcal{S}_h^{k,n}\Big)^{-1} K_s(., y),$$

where

$$(\mathcal{S}_h^{k,n})^* : L^2_{\mu_{k,n}}(\mathbb{R}^2) \longrightarrow W^s_{k,n}(\mathbb{R})$$

is the adjoint operator of $\mathcal{S}_h^{k,n}$ *given by*

$$\langle \mathcal{S}_h^{k,n}(f), g\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = \langle f, (\mathcal{S}_h^{k,n})^* g\rangle_{W^s_{k,n}(\mathbb{R})}, \quad f \in W^s_{k,n}(\mathbb{R}),\ g \in L^2_{\mu_{k,n}}(\mathbb{R}^2).$$

Moreover, for all $y \in \mathbb{R}$ *we have*

(i) $||\mathfrak{K}^{k,n}_{r,h}(., y)||_{W^s_{k,n}(\mathbb{R})} \leqslant \frac{C(k,n,s)}{r}$,

(ii) $||\mathcal{S}_h^{k,n}\big(\mathfrak{K}^{k,n}_{r,h}(., y)\big)||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant \frac{C(k,n,s)}{\sqrt{2r}}$,

(iii) $||(\mathcal{S}_h^{k,n})^*\mathcal{S}_h^{k,n}\big(\mathfrak{K}^{k,n}_{r,h}(., y)\big)||_{W^s_{k,n}(\mathbb{R})} \leqslant C(k, n, s)$*, where* $C(k, n, s)$ *is the constant given by (5.6).*

Proof. We proceed as above we prove that $(W^s_{k,n}(\mathbb{R}), \langle ., .\rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})})$ has a reproducing kernel denoted by $\mathfrak{K}^{k,n}_{r,h}$. On the other hand, we have

$$\begin{aligned} f(y) &= \langle f, \mathfrak{K}^{k,n}_{r,h}(., y)\rangle_{\mathcal{S}_h^{k,n},r,W^s_{k,n}(\mathbb{R})} \\ &= r\langle f, \mathfrak{K}^{k,n}_{r,h}(., y)\rangle_{W^s_{k,n}(\mathbb{R})} + \langle \mathcal{S}_h^{k,n}(f), \mathcal{S}_h^{k,n}(\mathfrak{K}^{k,n}_{r,h}(., y))\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &= \langle f, (rI + (\mathcal{S}_h^{k,n})^*\mathcal{S}_h^{k,n})\mathfrak{K}^{k,n}_{r,h}(., y)\rangle_{W^s_{k,n}(\mathbb{R})}. \end{aligned}$$

Thus,

$$(rI + (\mathcal{S}_h^{k,n})^*\mathcal{S}_h^{k,n})\mathfrak{K}^{k,n}_{r,h}(., y) = K_s(., y). \tag{5.16}$$

Furthermore, the previous identity implies that

$$\begin{aligned}&r^2||\mathfrak{K}^{k,n}_{r,h}(.,y)||^2_{W^s_{k,n}(\mathbb{R})}+2r||\mathcal{S}^{k,n}_h(\mathfrak{K}^{k,n}_{r,h}(.,y))||^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\\&+||(\mathcal{S}^{k,n}_h)^*\mathcal{S}^{k,n}_h(\mathfrak{K}^{k,n}_{r,h}(.,y))||^2_{W^s_{k,n}(\mathbb{R})}\\&=||K_s(.,y)||^2_{W^s_{k,n}(\mathbb{R})}.\end{aligned}$$

From this relation and using the fact that

$$||K_s(.,y)||_{W^s_{k,n}(\mathbb{R})}\leqslant C(k,n,s),$$

we obtain the properties (i), (ii), and (iii). □

Remark 5.6. *Using similar ideas as in Proposition 5.1, we prove*

$$\mathfrak{K}^{k,n}_{r,h}(x,y)=\int_{\mathbb{R}}\frac{B_{k,n}((-1)^n\xi,x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s+C_h}d\gamma_{k,n}(\xi). \tag{5.17}$$

We can now state the main result of this paragraph.

Theorem 5.3. *Let $s>\frac{(2k-1)n+2}{2n}$ and h be in $L^2_{k,n,e}(\mathbb{R})$.*

i*) For any $g\in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ and for any $r>0$, the best approximate function $f^*_{r,g}$ in the sense*

$$\begin{aligned}&\inf_{f\in W^s_{k,n}(\mathbb{R})}\left\{r\|f\|^2_{W^s_{k,n}(\mathbb{R})}+\|g-\mathcal{S}^{k,n}_h(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right\}=\\&r\|f^*_{r,g}\|^2_{W^s_{k,n}(\mathbb{R})}+\|g-\mathcal{S}^{k,n}_h(f^*_{r,g})\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\end{aligned} \tag{5.18}$$

exists uniquely and $f^*_{r,g}$ is defined by

$$f^*_{r,g}(y)=\langle g,\mathcal{S}^{k,n}_h\big(\mathfrak{K}^{k,n}_{r,h}(.,y)\big)\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

ii*) The best approximate function $f^*_{r,g}$ is represented by*

$$f^*_{r,g}(y)=\int_{\mathbb{R}^2}g(\nu,x)\mathfrak{Q}^{k,n}_{r,h}(\nu,x,y)d\mu_{k,n}(\nu,x),$$

where

$$\mathfrak{Q}^{k,n}_{r,h}(\nu,x,y)=\int_{\mathbb{R}}\frac{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)B_{k,n}((-1)^n\xi,x)B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s+C_h}d\gamma_{k,n}(\xi).$$

(iii) The extremal function $f^*_{r,g}$ satisfies the following inequality:

$$\forall\, y\in\mathbb{R},\quad |f^*_{r,g}(y)|\leqslant\frac{C(k,n,s)}{\sqrt{2r}}||g||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

Proof. (i) The existence and uniqueness of extremal function $f^*_{r,g}$ satisfying (5.18) is given by [33], and the extremal function $f^*_{r,g}$ is represented by

$$f^*_{r,g}(y)=\langle g,\mathcal{S}^{k,n}_h(\mathfrak{K}^{k,n}_{r,h}(.,y))\rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},\quad y\in\mathbb{R}.$$

(ii) Involving Lemma 4.1 we have for all $x\in\mathbb{R}$

$$\mathcal{S}^{k,n}_h(f)(x,\nu)=\int_{\mathbb{R}}\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu\Delta_\nu h})(\xi)\mathcal{F}_{k,n}(f)(\xi)B_{k,n}((-1)^n\xi,x)d\gamma_{k,n}(\xi).$$

Using the properties of the kernel $\mathfrak{K}_{r,h}^{k,n}$ and the definition of the deformed Stockwell transform, we get

$$\begin{aligned}\mathcal{S}_h^{k,n}\Big(\mathfrak{K}_{r,h}^{k,n}(.,y)\Big)(x,\nu) &= \int_{\mathbb{R}} \frac{\mathcal{F}_{k,n}(\overline{\mathcal{M}_\nu \Delta_\nu h})(\xi) B_{k,n}((-1)^n\xi,x) B_{k,n}(\xi,y)}{r(1+|\xi|^2)^s + C_h} d\gamma_{k,n}(\xi)\\ &= \mathfrak{Q}_{r,h}^{k,n}(\nu,x,y).\end{aligned}$$

This gives the result.

(iii) From Proposition 5.5 (ii), we have

$$|f^*_{r,g}(y)| \leqslant ||g||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} ||\mathcal{S}_h^{k,n}(\mathfrak{K}_{r,h}^{k,n}(.,y))||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant \frac{C(k,n,s)}{\sqrt{2r}} ||g||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.$$

Thus the theorem is proved. □

Corollary 5.3. *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$. If f is in $W^s_{k,n}(\mathbb{R})$ and $g = \mathcal{S}_h^{k,n}(f)$. Then*

(i) $\{f^*_{r,g}\}_{r>0}$ converges uniformly to f as $r \to 0^+$.

(ii) For all $y \in \mathbb{R}$, $f(y) = \lim_{r\to 0^+} f^*_{r,g}(y)$.

(iii) For all $y \in \mathbb{R}$, $|f(y) - f^*_{r,g}(y)| \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}$.

(iv) For all $y \in \mathbb{R}$, $|f^*_{r,g}(y)| \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}$.

Proof. Let f be in $W^s_{k,n}(\mathbb{R})$.

(i) Then

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,g}(y) = \langle f, (\mathcal{S}_h^{k,n})^* \mathcal{S}_h^{k,n}(\mathfrak{K}_{r,h}^{k,n}(.,y))\rangle_{W^s_{k,n}(\mathbb{R})}. \tag{5.19}$$

But from (5.16), we have

$$\forall\, y \in \mathbb{R}, \quad \lim_{r\to 0^+} (\mathcal{S}_h^{k,n})^* \mathcal{S}_h^{k,n}(\mathfrak{K}_{r,h}^{k,n}(.,y)) = K_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f^*_{r,g}(y) = \langle f, K_s(.,y)\rangle_{W^s_{k,n}(\mathbb{R})} = f(y).$$

(ii) From (5.16) and (5.19), the extremal function $f^*_{r,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad f^*_{r,g}(y) = f(y) - r\langle f, \mathfrak{K}_{r,h}^{k,n}(.,y)\rangle_{W^s_{k,n}(\mathbb{R})}.$$

Thus by Proposition 5.5 (i) we obtain, for all $y \in \mathbb{R}$

$$|f^*_{r,g}(y) - f(y)| \leqslant r||f||_{W^s_{k,n}(\mathbb{R})} ||\mathfrak{K}_{r,h}^{k,n}(.,y)||_{W^s_{k,n}(\mathbb{R})} \leqslant C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}.$$

(iii) From (5.19) and Proposition 5.5 (iii), the extremal function $f^*_{r,g}$ satisfies

$$\forall\, y \in \mathbb{R}, \quad \begin{array}{rcl}|f^*_{r,g}(y)| &\leqslant& ||f||_{W^s_{k,n}(\mathbb{R})} ||(\mathcal{S}_h^{k,n})^* \mathcal{S}_h^{k,n}(\mathfrak{K}_{r,h}^{k,n}(.,y))||_{W^s_{k,n}(\mathbb{R})}\\ &\leqslant& C(k,n,s)||f||_{W^s_{k,n}(\mathbb{R})}\end{array}.$$

□

Remark 5.7. *Let $s > \frac{(2k-1)n+2}{2n}$ and $r > 0$.*

If $\mathcal{S}_h^{k,n}$ is isometry (i.e. $(\mathcal{S}_h^{k,n})^* \mathcal{S}_h^{k,n} = Id$) then

(i) $\langle .,.\rangle_{\mathcal{S}_h^{k,n}, r, W^s_{k,n}(\mathbb{R})} = (r+1)\langle .,.\rangle_{W^s_{k,n}(\mathbb{R})}$.

(ii) $\mathfrak{K}_{r,h}^{k,n}(x,y) = \frac{1}{r+1} K_s(x,y)$, for all $x, y \in \mathbb{R}$.

(iii) For all $y\in\mathbb{R}$, $f^*_{r,g}(y)=\frac{1}{r+1}(\mathcal{S}^{k,n}_h)^*g(y)$, $g\in L^2_{\mu_{k,n}}(\mathbb{R}^2)$.
(iv) For all $y\in\mathbb{R}$, $f^*_{r,\mathcal{S}^{k,n}_h(u)}(y)=\frac{1}{r+1}u(y)$, $u\in W^s_{k,n}(\mathbb{R})$.

Remark 5.8. *One of our motivations for introducing the theory of reproducing kernels for the best approximation problems involving the deformed Stockwell transform is to push forward the connection between Stockwell analysis and numerical analysis. We think of the results presented in this Section as opening potentially interesting studies by using computers and graphs, to illustrate numerical experiments approximation formulas for the limit case* $r\uparrow 0$.

6. TIME-FREQUENCY CONCENTRATION FOR $\mathcal{S}^{k,n}_h$

6.1. Weighted inequalities for $\mathcal{S}^{k,n}_h$.

Proposition 6.1. *Let* h *be a deformed Hankel wavelet on* $\mathbb{R}$ *in* $L^1_{k,n}(\mathbb{R})\cap L^2_{k,n}(\mathbb{R})$. *Then,* $\mathcal{S}^{k,n}_h(L^2_{k,n}(\mathbb{R}))$ *is a reproducing kernel Hilbert space with kernel function*

$$\mathcal{K}_h(x',\nu';x,\nu):=\frac{1}{C_h}\int_{\mathbb{R}}h_{x',\nu'}(y)\overline{h_{x,\nu}(y)}d\gamma_{k,n}(y). \tag{6.1}$$

The kernel $\mathcal{K}_h$ *satisfies :*

$$\forall\,(x',\nu'),\,(x,\nu)\in\mathbb{R}^2,\quad |\mathcal{K}_h(x',\nu';x,\nu)|\leqslant\frac{||h||^2_{L^2_{k,n}(\mathbb{R})}}{C_h} \tag{6.2}$$

Proof. Let f be in $L^2_{k,n}(\mathbb{R})$. We have

$$\mathcal{S}^{k,n}_h(f)(x,\nu)=\int_{\mathbb{R}}f(y)\overline{h_{x,\nu}(y)}d\gamma_{k,n}(y),\ (x,\nu)\in\mathbb{R}^2.$$

Using relation (4.19), we obtain

$$\mathcal{S}^{k,n}_h(f)(x,\nu)=\frac{1}{C_h}\int_{\mathbb{R}}\int_{\mathbb{R}}\mathcal{S}^{k,n}_h(f)(x',\nu')\overline{\mathcal{S}^{k,n}_h(h_{x,\nu})(x',\nu')}d\mu_{k,n}(x',\nu').$$

On the other hand, using Proposition 3.5 (i), one can easily see that the function

$$x'\mapsto\frac{1}{C_h}\overline{\mathcal{S}^{k,n}_h(h_{x,\nu})(x',\nu')}=\frac{1}{C_h}\int_{\mathbb{R}}h_{x',\nu'}(y)\overline{h_{x,\nu}(y)}d\gamma_{k,n}(y)$$

belongs to $L^2_{k,n}(\mathbb{R})$, for every $(x,\nu),(x',\nu')\in\mathbb{R}^2$. Therefore, the result is obtained. □

In order to prove a concentration result of the continuous deformed Stockwell transform, we need the following notations:

$P_U: L^2_{\mu_{k,n}}(\mathbb{R}^2)\to L^2_{\mu_{k,n}}(\mathbb{R}^2)$ the orthogonal projection from $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ onto the subspace of function supported in the subset $U\subset\mathbb{R}^2$ with

$$0<\mu_{k,n}(U):=\int_U d\mu_{k,n}(x,\nu)<\infty.$$

$P_h: L^2_{\mu_{k,n}}(\mathbb{R}^2)\to L^2_{\mu_{k,n}}(\mathbb{R}^2)$ the orthogonal projection from $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ onto $\mathcal{S}^{k,n}_h(L^2_{k,n}(\mathbb{R}))$.

We put

$$||P_U P_h|| := \sup\Big\{||P_U P_h v||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} :\ v \in L^2_{\mu_{k,n}}(\mathbb{R}^2),\ ||v||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = 1\Big\}.$$

In the following we will prove the concentration of $\mathcal{S}_h^{k,n}(f)$ in small sets.

Proposition 6.2. *Let h be in $L^2_{k,n,e}(\mathbb{R})$ and $U \subset \mathbb{R}^2$ with*

$$0 < \mu_{k,n}(U) < \frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}}.$$

Then, for all $f \in L^2_{k,n}(\mathbb{R})$ we have

$$\|\mathcal{S}_h^{k,n}(f) - \chi_U \mathcal{S}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \sqrt{C_h - \mu_{k,n}(U)\|h\|^2_{L^2_{k,n}(\mathbb{R})}}\,\|f\|_{L^2_{k,n}(\mathbb{R})}, \quad (6.3)$$

where χ_U denotes the characteristic function of U.

Proof. From Plancherel's formula (4.20) we have

$$C_h||f||^2_{L^2_{k,n}(\mathbb{R})} = \|\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} = \|\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(U)} + \|\mathcal{S}_h^{k,n}(f)\|^2_{L^2_{\mu_{k,n}}(U^c)}. (6.4)$$

On the other hand from the relation (4.16) we have

$$\begin{aligned}\int_U |\mathcal{S}_h^{k,n}(f)(x,\nu)|^2 d\mu_{k,n}(x,\nu) &\leqslant \|\mathcal{S}_h^{k,n}(f)\|^2_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)}\mu_{k,n}(U) \\ &\leqslant \mu_{k,n}(U)\|f\|^2_{L^2_{k,n}(\mathbb{R})}\|h\|^2_{L^2_{k,n}(\mathbb{R})}.\end{aligned} \quad (6.5)$$

Thus the result follows immediately by integrating (6.4) in (6.5). □

Remark 6.1. *We assume that $0 < \mu_{k,n}(U) < \frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}}$. If $\mathcal{S}_h^{k,n}(f)$ is supported in U, then $f = 0$.*

Proposition 6.3. *Let h be in $L^2_{k,n,e}(\mathbb{R})$.*

Let $s > 0$. Then the following uncertainty inequalities hold.

(1) A Heisenberg-type uncertainty inequalities for $\mathcal{S}_h^{k,n}$:

(i) There exists a constant $C_1(k,n,s,h) > 0$ such that, for all f in $L^2_{k,n}(\mathbb{R})$, we have

$$\Big|\Big|\, ||(x,\nu)||^s \mathcal{S}_h^{k,n}(f)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C_1(k,n,s,h)\|f\|_{L^2_{k,n}(\mathbb{R})}. \quad (6.6)$$

(ii) There exists a constant $C_2(k,n,s,h) > 0$ such that, for all f in $L^2_{k,n}(\mathbb{R})$, we have

$$\Big|\Big|\, |x|^s \mathcal{S}_h^{k,n}(f)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \Big|\Big|\, |\nu|^s \mathcal{S}_h^{k,n}(f)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C_2(k,n,s,h)\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \quad (6.7)$$

(2) A Faris local uncertainty inequality for $\mathcal{S}_h^{k,n}$:

There exists a constant $C_3(k,n,s,h) > 0$ such that, for all f in $L^2_{k,n}(\mathbb{R})$, and every subset $U \subset \mathbb{R}^2$ such that $0 < \mu_{k,n}(U) < \infty$, we have

$$||\mathcal{S}_h^{k,n}(f)||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant C_3(k,n,s,h)\sqrt{\mu_{k,n}(U)}\Big|\Big|\, ||(x,\nu)||^s \mathcal{S}_h^{k,n}(f)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \quad (6.8)$$

Proof. (1) Let $r > 0$ such that $0 < \mu_{k,n}(B_2(0,r)) < \frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}}$ where $B_2(0,r)$ is the open ball of $\mathbb{R}^2$ defined by

$$B_2(0,r) = \Big\{(x,\nu) \in \mathbb{R}^2 : ||(x,\nu)|| < r\Big\}.$$

Involving the relation (6.3) with $U = B_2(0,r)$, and by simple calculation we obtain

$$\begin{aligned}\big(C_h - \mu_{k,n}(U)\|h\|^2_{L^2_{k,n}(\mathbb{R})}\big)||f||^2_{L^2_{k,n}(\mathbb{R})} &\leqslant \int_{B_2(0,r)^c} |\mathcal{S}_h^{k,n}(f)(x,\nu)|^2 d\mu_k(x,\nu)\\ &\leqslant \tfrac{1}{r^{2s}}\Big|\Big|\, ||(x,\nu)||^s \mathcal{S}_h^{k,n}(f)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

Thus we obtain (6.6) with $C_1(k,n,s,h) := r^s\sqrt{C_h - \mu_{k,n}(U)\|h\|^2_{L^2_{k,n}(\mathbb{R})}}$.

(ii) By applying the inequality $||(x,\nu)||^s \leqslant 2^s(|\nu|^s + |x|^s)$ in (6.6), we get

$$\Big|\Big|\, |x|^s \mathcal{S}_h^{k,n}(f)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + ||\,|\nu|^s\mathcal{S}_h^{k,n}(f)||^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \frac{\big(C_1(k,n,s)\big)^2}{2^{2s}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}. \quad (6.9)$$

We replace f by f_t, in the relation (6.9), we apply (4.17) and next we make a change of variables in each term, we obtain the following relation:

$$t^{2s}||\,|x|^s\mathcal{S}_h^{k,n}(f)||^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + t^{-2s}||\,|\nu|^s\mathcal{S}_h^{k,n}(f)||^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \frac{\big(C_1(k,n,s)\big)^2}{2^{2s}}\|f\|^2_{L^2_{k,n}(\mathbb{R})}.$$

Then (6.7) follows by minimizing the left hand side of this inequality, with respect $t > 0$.

(2) Using the fact that

$$||\mathcal{S}_h^{k,n}(f)||_{L^2_{\mu_{k,n}}(U)} \leqslant \sqrt{\mu_{k,n}(U)}||\mathcal{S}_h^{k,n}(f)||_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)},$$

and the fact that

$$||\mathcal{S}_h^{k,n}(f)||_{L^\infty_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant ||h||_{L^2_{k,n}(\mathbb{R})}||f||_{L^2_{k,n}(\mathbb{R})},$$

then we get

$$||\mathcal{S}_h^{k,n}(f)||_{L^2_{\mu_{k,n}}(U)} \leqslant \sqrt{\mu_{k,n}(U)}||h||_{L^2_{k,n}(\mathbb{R})}||f||_{L^2_{k,n}(\mathbb{R})}.$$

Finally, we obtain the result from (6.6). □

6.2. Benedicks-type uncertainty principle for $\mathcal{S}_h^{k,n}$.

In the following we will prove the concentration of $\mathcal{S}_h^{k,n}(f)$ in arbitrary sets of finite measures.

Theorem 6.1. *Let h be in $L^2_{k,n,e}(\mathbb{R})$ and $U \subset \mathbb{R}^2$ with $0 < \mu_{k,n}(U) < \infty$.*

If $P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)) = \{0\}$ Then, there exists a positive constant $C := C_{k,n}(h,U)$ such that for all $f \in L^2_{k,n}(\mathbb{R})$, we have

$$\|\mathcal{S}_h^{k,n}(f) - \chi_U\mathcal{S}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq C\|f\|_{L^2_{k,n}(\mathbb{R})}. \quad (6.10)$$

For the proof of this theorem, we need the following lemma.

Lemma 6.1. *([40]). Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two closed subspaces of a Hilbert space $\mathcal{H}$ satisfying $\mathcal{H}_1 \bigcap \mathcal{H}_2 = \{0\}$. Let $P_{\mathcal{H}_1}$ and $P_{\mathcal{H}_2}$ denote the corresponding orthogonal projections, and assume the product $P_{\mathcal{H}_1}P_{\mathcal{H}_2}$ to be a compact operator. Then, there exists a constant $C>0$ such that for $f\in\mathcal{H}$*

$$||P_{\mathcal{H}_1^\perp}f||_{\mathcal{H}} + ||P_{\mathcal{H}_2^\perp}f||_{\mathcal{H}} \geq C||f||_{\mathcal{H}}. \tag{6.11}$$

Proof. of Theorem 6.1. Defining $\mathcal{H}_1$ and $\mathcal{H}_2$ by

$$\mathcal{H}_1 := P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)), \quad \mathcal{H}_2 := P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)).$$

We proceed as in [21], we prove that

$$\begin{aligned} ||P_UP_h||^2_{HS}: &= \int_{\mathbb{R}^2\times\mathbb{R}^2} |\chi_U(x,\nu)|^2|\mathcal{K}_h(\nu',x';\nu,x)|^2 d\mu_{k,n}(x',\nu')d\mu_{k,n}(x,\nu) \\ &\leqslant \frac{||h||^2_{L^2_{k,n}(\mathbb{R})}}{C_h}\mu_{k,n}(U) < \infty. \end{aligned} \tag{6.12}$$

Hence, P_UP_h is a Hilbert-Schmidt operator and, therefore, compact. Now, Lemma 6.1 implies the existence of a constant $C>0$ such that (6.11) holds for $P_{\mathcal{H}_1} := P_U$ and $P_{\mathcal{H}_2} := P_h$. Since

$$P_{\mathcal{H}_2^\perp}(\mathcal{S}_h^{k,n}(f)) = (Id - P_h)\mathcal{S}_h^{k,n}(f) = 0,$$

this leads to (6.10). □

Definition 6.1. *Let h be in $L^2_{k,n,e}(\mathbb{R})$ and $U\subset\mathbb{R}^2$ such that $0<\mu_{k,n}(U)<\infty$.*

(1) We say that U is weakly annihilating, if any function $f\in L^2_{k,n}(\mathbb{R})$ vanishes when its deformed Stockwell transform $\mathcal{S}_h^{k,n}(f)$ is supported in U.

(2) We say that U is strongly annihilating, if there exists a positive constant $\mathfrak{C}_{k,n}(U,h)$ such that for every function $f\in L^2_{k,n}(\mathbb{R})$,

$$\|\mathcal{S}_h^{k,n}(f) - \chi_U\mathcal{S}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \mathfrak{C}_{k,n}(U,h)\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{6.13}$$

The constant $\mathfrak{C}_{k,n}(U,h)$ will be called the annihilation constant of U.

Remark 6.2. *(1) It is clear that, every strongly annihilating set is also a weakly.*

(2) From Proposition 6.2, we see that any set $U\subset\mathbb{R}^2$ with

$$0<\mu_{k,n}(U)<\frac{C_h}{\|h\|^2_{L^2_{k,n}(\mathbb{R})}},$$

is strongly annihilating.

(3) As the operator P_UP_h is Hilbert-Schmidt hence is compact, then from [15] we have if U is weakly annihilating, it is also strongly annihilating.

(4) If $||P_UP_h||<1$, then for all $f\in L^2_{k,n}(\mathbb{R})$

$$\frac{1}{\sqrt{1-||P_UP_h||^2}}\|\mathcal{S}_h^{k,n}(f) - \chi_U\mathcal{S}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \|f\|_{L^2_{k,n}(\mathbb{R})}\|h\|_{L^2_{k,n}(\mathbb{R})}. \tag{6.14}$$

(5) Following the result established in a general context in [15] p.88, we have if U is strongly annihilating, then $||P_U P_h|| < 1$.

In the next, we give the Benedicks-type uncertainty principle for the deformed Stockwell transform.

Theorem 6.2. *We suppose that the deformed Stockwell wavelet h satisfies*

$$\int_{\{\xi\in\mathbb{R}:\tau_1^{k,n}(\mathcal{F}_{k,n}(h))(\xi)\neq 0\}} d\gamma_{k,n}(\xi) < \infty. \tag{6.15}$$

Then for any subset $U \subset \mathbb{R}^2$ such that for almost every $\nu \in \mathbb{R}$,

$$\int_{\mathbb{R}} \chi_{_U}(x,\nu) d\gamma_{k,n}(x) < \infty, \tag{6.16}$$

where $\chi_{_U}$ denotes the characteristic function of U, we have

$$P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2)) = \{0\}. \tag{6.17}$$

Proof. Let F be a non-trivial function in $P_h(L^2_{\mu_{k,n}}(\mathbb{R}^2)) \cap P_U(L^2_{\mu_{k,n}}(\mathbb{R}^2))$, then there exists a function $f \in L^2_{k,n}(\mathbb{R})$ such that $F = \mathcal{S}_h^{k,n}(f)$ and $supp\, F \subset U$.
Let $\nu \in \mathbb{R}$, such that $\int_{\mathbb{R}} \chi_{_U}(x,\nu) d\gamma_{k,n}(x) < \infty$. Consider the function $\mathcal{S}_h^{k,n}(f)(x,\nu)$ with respect to the variable x, we denote it $F_\nu(x)$. Thus, we have

$$supp\, F_\nu \subset \Big\{x \in \mathbb{R} : (x,\nu) \in U\Big\}$$

and

$$\int_{supp\, F_\nu} d\gamma_{k,n}(x) < \infty.$$

On the other hand using (4.18) and the hypothesis (6.15), we get

$$\int_{\{\xi\in\mathbb{R}:\mathcal{F}_{k,n}(F_\nu)(\xi)\neq 0\}} d\gamma_{k,n}(\xi) < \infty.$$

Using Proposition 2.4, we deduce that for every $x \in \mathbb{R}$, $F_\nu(x) = 0$, which implies that $F = 0$. □

Consequently, we obtain the following improvement.

Corollary 6.1. *Let h be a deformed Stockwell wavelet satisfying the relation (6.15). Then for any subset $U \subset \mathbb{R}^2$ verifying the relation (6.16), there exists a constant $\mathfrak{C}_{k,n}(U,h) > 0$ such that for all f in $L^2_{k,n}(\mathbb{R})$, we have*

$$\|\chi_{_{U^c}} \mathcal{S}_h^{k,n}(f)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \geq \mathfrak{C}_{k,n}(U,h)\|f\|_{L^2_{k,n}(\mathbb{R})}. \tag{6.18}$$

6.3. Donoho-Stark's uncertainty inequality for $\mathcal{S}_h^{k,n}$.
Now we will derive a sufficient condition by means of which one can recover a signal F belongs to $L^2_{\mu_{k,n}}(\mathbb{R}^2)$ from the knowledge of a truncated version of it, following the Donoho-Stark criterion [9].

Let h be in $L^2_{k,n,e}(\mathbb{R})$. A signal $F \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ is transmitted to a receiver who knows that $F \in \mathcal{S}_h^{k,n}(L^2_{k,n}(\mathbb{R}))$. Suppose that the observation of F is corrupted by a noise $\mathcal{N} \in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ (which is nonetheless assumed to be small) and unregistered values on $U \in \mathbb{R}^2$. Thus, the observable function r satisfies

$$r(x,\nu) = \begin{cases} F(x,\nu) + \mathcal{N}(x,\nu) & \text{if} \quad (x,\nu) \in U^c \\ 0 & \text{if} \quad (x,\nu) \in U. \end{cases} \tag{6.19}$$

Here we have assumed without loss of generality that $\mathcal{N} = 0$ on U. Equivalently,

$$r = (Id - P_U)F + \mathcal{N}. \tag{6.20}$$

We say that F can be stably reconstructed from r, if there exists a linear operator

$$L_{U,h} : L^2_{\mu_{k,n}}(\mathbb{R}^2) \to L^2_{\mu_{k,n}}(\mathbb{R}^2)$$

and a constant $\mathcal{C}_{k,n}(U,h)$ such that

$$||F - L_{U,h}(r)||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant \mathcal{C}_{k,n}(U,h)||\mathcal{N}||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}. \tag{6.21}$$

Theorem 6.3. *Retain the assumption of Theorem 6.2. Then F can be stably reconstructed from r. Moreover, the constant $\mathcal{C}_{k,n}(U,h)$ in (6.21) is not larger than $(1 - ||P_U P_h||)^{-1}$.*

Proof. We apply the same arguments that used in [9]. From Corollary 6.1, U is strongly annihilating, then from Remark 6.2 we have $||P_U P_h|| < 1$. Therefore $I - P_U P_h$ is invertible. Let

$$L_{U,h} = (Id - P_U P_h)^{-1}.$$

As $F \in \mathcal{S}_h^{k,n}(L^2_{k,n}(\mathbb{R}))$, then $(I - P_U)F = (I - P_U P_h)F$. Thus by simple calculations we see that

$$F - L_{U,h} r = -L_{U,h}\mathcal{N}.$$

So that

$$\begin{aligned} ||F - L_{U,h} r||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &= ||L_{U,h}\mathcal{N}||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leqslant ||(Id - P_U P_h)^{-1}|| \, ||\mathcal{N}||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leqslant (1 - ||P_U P_h||)^{-1} \, ||\mathcal{N}||_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}, \end{aligned}$$

which allows to conclude. □

Remark 6.3. *(An algorithm for computing $L_{U,h} r$)*
The identity

$$L_{U,h} = (Id - P_U P_h)^{-1} = \sum_{j=0}^{\infty} (P_U P_h)^j$$

suggest an algorithm for computing $L_{U,h}r$. Using the similar method given in [9], we give an algorithm for computing $L_{U,h}r$. Indeed, put

$$F^{(m)} = \sum_{j=0}^{m} (P_U P_h)^j r,$$

then $F^{(m)} \to L_{U,h}(r)$ as $m \to \infty$. Now

$$\begin{array}{rcl} F^{(0)} & = & r \\ F^{(1)} & = & r + P_U P_h F^{(0)} \\ F^{(2)} & = & r + P_U P_h F^{(1)} \\ & \cdots & \end{array} \tag{6.22}$$

and so on. The iteration converges at a geometric rate to the fixed point

$$F = r + P_U P_h F.$$

Algorithms of type (6.22), have been applied to a host of problems in signal recovery see [9], and others.

7. SHAPIRO'S DISPERSION THEOREM FOR $\mathcal{S}_h^{k,n}$

In this section we will assume that h is a fixed function in $L^2_{k,n,e}(\mathbb{R})$ such that $C_h = 1$.

We denote by $B(L^2_{k,n}(\mathbb{R}))$, the space of bounded operators from $L^2_{k,n}(\mathbb{R})$ into itself.

Definition 7.1. *(i) The singular values $(s_j(A))_{j\in\mathbb{N}}$ of a compact operator A in $B(L^2_{k,n}(\mathbb{R}))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*

(ii) For $1 \leqslant p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{j=1}^{\infty} (s_j(A))^p \Big)^{\frac{1}{p}}. \tag{7.1}$$

(iii) We define $S_\infty := B(L^2_{k,n}(\mathbb{R}))$, equipped with the norm,

$$||A||_{S_\infty} := \sup_{v\in L^2_{k,n}(\mathbb{R}):||v||_{L^2_{k,n}(\mathbb{R})}=1} ||Av||_{L^2_{k,n}(\mathbb{R})}. \tag{7.2}$$

Definition 7.2. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{j=1}^{\infty} \langle Av_j, v_j \rangle_{L^2_{k,n}(\mathbb{R})} \tag{7.3}$$

where $(v_j)_j$ is any orthonormal basis of $L^2_{k,n}(\mathbb{R})$.

Remark 7.1. *If A is positive, then*

$$tr(A) = ||A||_{S_1}. \tag{7.4}$$

Moreover, a compact operator A on the Hilbert space $L^2_{k,n}(\mathbb{R})$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{j=1}^{\infty} ||Av_j||^2_{L^2_{k,n}(\mathbb{R})} \tag{7.5}$$

for any orthonormal basis $(v_j)_j$ of $L^2_{k,n}(\mathbb{R})$.

Definition 7.3. *Let $0 < \varepsilon < 1$ and $U \subset \mathbb{R}^2$ be a measurable subset. For f in $L^2_{k,n}(\mathbb{R})$, we say that $\mathcal{S}^{k,n}_h(f)$ is ε-concentrated on U if*

$$\left\| \mathcal{S}^{k,n}_h(f) \right\|_{L^2_{\mu_{k,n}}(U^c)} \leqslant \varepsilon \left\| \mathcal{S}^{k,n}_h(f) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},$$

where U^c is the complement of U in $\mathbb{R}^2$.

Proposition 7.1. *Let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$ and U be a measurable subset of $\mathbb{R}^2$ such that $0 < \mu_{k,n}(U) < \infty$. For every nonempty finite subset $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sum_{j\in\mathcal{E}} \left(1 - \left\| \mathbb{1}_{U^c} \mathcal{S}^{k,n}_h(\varphi_j) \right\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right) \leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} \mu_{k,n}(U).$$

Proof. Since $(\varphi_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, by (4.20) we deduce that $\left(\mathcal{S}^{k,n}_h(\varphi_j)\right)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{\mu_{k,n}}(\mathbb{R}^2)$. Moreover, since the operator $P_U P_h$ is of Hilbert-Schmidt type, then, by (7.5) and (7.3), it is easy to see that

$$\begin{aligned}\sum_{j\in\mathcal{E}} \langle P_U \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &= \sum_{j\in\mathcal{E}} \langle P_h P_U P_h \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leqslant \mathrm{tr}(P_h P_U P_h) \\ &= \|P_U P_h\|^2_{HS}.\end{aligned}$$

Further using (6.12), we get

$$\|P_U P_h\|_{HS} \leqslant ||h||_{L^2_{k,n}(\mathbb{R})} \sqrt{\mu_{k,n}(U)}.$$

Thus,

$$\sum_{j\in\mathcal{E}} \langle P_U \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} \mu_{k,n}(U). \tag{7.6}$$

On the other hand, by Cauchy-Schwarz's inequality we have for every $j \in \mathcal{E}$,

$$\begin{aligned}\langle P_U \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} &= 1 - \langle P_{U^c} \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\geqslant 1 - \|\mathbb{1}_{U^c} \mathcal{S}^{k,n}_h(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}.\end{aligned}$$

In particular, by relation (7.6), we obtain

$$\begin{aligned}\sum_{j\in\mathcal{E}} \left(1 - \|\mathbb{1}_{U^c} \mathcal{S}^{k,n}_h(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right) &\leqslant \sum_{j\in\mathcal{E}} \langle P_U \mathcal{S}^{k,n}_h(\varphi_j), \mathcal{S}^{k,n}_h(\varphi_j) \rangle_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \\ &\leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} \mu_{k,n}(U).\end{aligned}$$

□

Next, we shall use Proposition 7.1 to prove that if the deformed Stockwell transform of an orthonormal sequence is ε-concentrated on a given centered ball in $\mathbb{R}^2$, then a such sequence is necessary finite

Proposition 7.2. *Let ε and δ be two positive real numbers such that $0<\varepsilon<1$. Let $\mathcal{E}\subset\mathbb{N}$ be a nonempty subset and $(\varphi_j)_{j\in\mathcal{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. If, for every $j\in\mathcal{E}$, $\mathcal{S}_h^{k,n}(\varphi_j)$ is ε-concentrated on the ball*

$$B_2(0,\delta):=\Big\{(x,\nu)\in\mathbb{R}^2:||(x,\nu)||<\delta\Big\},$$

then the set $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E})\leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})}\frac{\big(\Gamma(\frac{(2k-1)n+2}{2n})\big)^2}{\Gamma(\frac{2kn+2}{n})(1-\varepsilon)}\delta^{\frac{2(2k-1)n+4}{n}}. \tag{7.7}$$

Proof. Let $\mathcal{M}\subset\mathcal{E}$ be a nonempty finite subset, then by Proposition 7.1, we deduce that

$$\sum_{n\in\mathcal{M}}\Big(1-\|\mathbb{1}_{B_2(0,\delta)^c}\mathcal{S}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\Big)\leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})}\mu_{k,n}(B_2(0,\delta)). \tag{7.8}$$

However, for every $j\in\mathcal{M}$, we have

$$\|\mathbb{1}_{B_2(0,\delta)^c}\mathcal{S}_h^{k,n}(\varphi_j)\|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\leqslant\varepsilon\,\text{and}\,\mu_{k,n}(B_2(0,\delta))=\frac{\big(\Gamma(\frac{(2k-1)n+2}{2n})\big)^2}{\Gamma(\frac{2kn+2}{n})}\delta^{\frac{2(2k-1)n+4}{n}}. \tag{7.9}$$

Hence, by combining relations (7.8) and (7.9), we deduce that

$$\mathrm{Card}(\mathcal{M})\leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})}\frac{\big(\Gamma(\frac{(2k-1)n+2}{2n})\big)^2}{\Gamma(\frac{2kn+2}{n})(1-\varepsilon)}\delta^{\frac{2(2k-1)n+4}{n}},$$

which means that $\mathcal{E}$ is finite and satisfies relation (7.7). □

For a positive real number p, the generalized p^{th} time-frequency dispersion of $\mathcal{S}_h^{k,n}(f)$ is defined by

$$\rho_p(\mathcal{S}_h^{k,n}(f)):=\left(\int_{\mathbb{R}^2}||(x,\nu)||^p\left|\mathcal{S}_h^{k,n}(f)(x,\nu)\right|^2d\mu_{k,n}(x,\nu)\right)^{\frac{1}{p}}.$$

Corollary 7.1. *Let A and p be two positive real numbers. Let $\mathcal{E}\subset\mathbb{N}$ be a nonempty subset and $(\varphi_j)_{j\in\mathcal{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. Assume that for every $j\in\mathcal{E}$,*

$$\rho_p(\mathcal{S}_h^{k,n}(\varphi_j))\leqslant A.$$

Then $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E})\leqslant M(k,n,p)||h||^2_{L^2_{k,n}(\mathbb{R})}A^{\frac{2(2k-1)n+4}{n}},$$

where

$$M(k,n,p)=2^{\frac{8kn+(p-4)n+8}{np}}\frac{\big(\Gamma(\frac{(2k-1)n+2}{2n})\big)^2}{\Gamma(\frac{2kn+2}{n})}.$$

Proof. Since $\rho_p(\mathcal{S}_h^{k,n}(\varphi_j)) \leqslant A$ for every $j \in \mathcal{E}$, it follows

$$\int_{B_2^c(0,A2^{\frac{2}{p}})} |\mathcal{S}_h^{k,n}(\varphi_j)(x,\nu)|^2 d\mu_{k,n}(x,\nu) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p} \rho_p^p(\mathcal{S}_h^{k,n}(\varphi_j)) \leqslant \frac{1}{4}. \tag{7.10}$$

The inequality (7.10) means that for every $j \in \mathcal{E}$, $\mathcal{S}_h^{k,n}(\varphi_j)$ is $\frac{1}{2}$-concentrated in the ball $B_2(0, A2^{\frac{2}{p}})$. According to Proposition 7.2, we deduce that $\mathcal{E}$ is finite and

$$\mathrm{Card}(\mathcal{E}) \leqslant M(k,n,p)||h||^2_{L^2_{k,n}(\mathbb{R})} A^{\frac{2(2k-1)n+4}{n}}.$$

□

Lemma 7.1. *Let p be a positive real number. If $(\varphi_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\rho_p^p(\mathcal{S}_h^{k,n}(\varphi_j)) \geqslant 2^{p(j_0-1)}, \qquad \forall j \in \mathbb{N}.$$

Proof. Proceeding as in [28], using the assumptions $C_h = 1$ and the fact that $(\varphi_j)_{j\in\mathbb{N}}$ is an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$, we infer that there exist a positive constant $C(k,n,p)$ such that

$$\rho_p^p(\mathcal{S}_h^{k,n}(\varphi_j)) \geqslant \frac{1}{\left(C(k,n,p)\right)^2}.$$

Moreover it is easy to see that there exists $j_0 \in \mathbb{Z}$ such that

$$\frac{1}{\left(C(k,n,p)\right)^2} \geq 2^{p(j_0-1)}.$$

Thus the desired result is proved. □

Theorem 7.1 (Shapiro's dispersion theorem for $\mathcal{S}_h^{k,n}$)**.** *Let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. For every positives reals numbers p, there is a positive constant C such that for every nonempty finite subset $\mathcal{E} \subset \mathbb{N}$, we have*

$$\sum_{j\in\mathcal{E}} (\rho_p(\mathcal{S}_h^{k,n}(\varphi_j)))^p \geqslant \frac{1}{2}\left(\frac{3||h||^{-2}_{L^2_{k,n}(\mathbb{R})}}{2^{\frac{8kn-3n+8}{n}} M(k,n,p)}\right)^{\frac{np}{2(2k-1)n+4}} \left(\mathrm{Card}(\mathcal{E})\right)^{1+\frac{np}{2(2k-1)n+4}}. \tag{7.11}$$

Proof. For every $j \in \mathbb{Z}$, let

$$P_j = \left\{ m \in \mathbb{N} \ : \ \rho_p(\mathcal{S}_h^{k,n}(\varphi_m)) \in [2^{j-1}, 2^j) \right\}.$$

Then, for every $m \in P_j$,

$$\int_{\mathbb{R}^2} ||(x,\nu)||^p \left|\mathcal{S}_h^{k,n}(\varphi_m)(x,\nu)\right|^2 d\mu_{k,n}(x,\nu) \leqslant 2^{jp}.$$

That is the sequence $(\varphi_m)_{m\in P_j}$ satisfies the conditions of Corollary 7.1, and therefore P_j is finite with

$$\mathrm{Card}(P_j) \leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} M(k,n,p) 2^{(\frac{2(2k-1)n+4}{n})j}. \tag{7.12}$$

For $m \in \mathbb{Z},\ m \geqslant j_0$, we denote by $Q_m := \bigcup_{j=j_0}^{m} P_j$. According to (7.12), we have

$$\mathrm{Card}\,(Q_m) = \sum_{j=j_0}^{m} \mathrm{Card}(P_j) \leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} \frac{M(k,n,p)2^{\frac{2(2k-1)n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})m}.$$

Now, if $\mathrm{Card}(\mathcal{E}) > ||h||^2_{L^2_{k,n}(\mathbb{R})} \dfrac{M(k,n,p)2^{\frac{4kn-n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})j_0}$, then we can choose an integer $m > j_0$ such that

$$\begin{aligned} \frac{M(k,n,p)2^{\frac{4kn-n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})(m-1)} &< \frac{\mathrm{Card}(\mathcal{E})}{||h||^2_{L^2_{k,n}(\mathbb{R})}} \\ &\leqslant \frac{M(k,n,p)2^{\frac{4kn-n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})m}. \end{aligned} \tag{7.13}$$

Thus, by (7.13), we get

$$\begin{aligned} \sum_{j\in\mathcal{E}} \Big(\rho_p(\mathcal{S}_h^{k,n}(\varphi_j))\Big)^p &\geqslant \frac{\mathrm{Card}(\mathcal{E})}{2} 2^{(m-1)p} \\ &\geqslant \tfrac{1}{2}\big(\mathrm{Card}(\mathcal{E})\big)^{1+\frac{np}{2(2k-1)n+4}} \left(\frac{3}{||h||^2_{L^2_{k,n}(\mathbb{R})} 2^{\frac{8kn-3n+8}{n}} M(k,n,p)}\right)^{\frac{np}{2(2k-1)n+4}}. \end{aligned}$$

Finally, if $\mathrm{Card}(\mathcal{E}) \leqslant ||h||^2_{L^2_{k,n}(\mathbb{R})} \dfrac{M(k,n,p)2^{\frac{4kn-n+4}{n}}}{3} 2^{(\frac{2(2k-1)n+4}{n})j_0}$, then

$$\begin{aligned} \sum_{j\in\mathcal{E}} \Big(\rho_p(\mathcal{S}_h^{k,n}(\varphi_j))\Big)^p &\geqslant \mathrm{Card}(\mathcal{E}) 2^{(j_0-1)p} \\ &\geqslant \big(\mathrm{Card}(\mathcal{E})\big)^{1+\frac{p}{4k}} \left(\frac{3||h||^{-2}_{L^2_{k,n}(\mathbb{R})}}{2^{\frac{8kn-3n+8}{n}} M(k,n,p)}\right)^{\frac{np}{2(2k-1)n+4}}. \end{aligned}$$

□

Remark 7.2. *By taking* $\mathrm{Card}(\mathcal{E}) = 1$*, relation (7.11) appears as a general version of Heisenberg-Pauli-Weyl inequality for the deformed Stockwell transform including the* p^{th} *dispersion.*

Corollary 7.2. *Let* $p > 0$ *and let* $(\varphi_j)_{j\in\mathbb{N}}$ *be an orthonormal sequence in* $L^2_{k,n}(\mathbb{R})$*. Then for every* $\mathcal{E} \subset \mathbb{N}$

$$\begin{aligned} &\sum_{j\in\mathcal{E}} \left(\Big\| |\nu|^{\frac{p}{2}} \mathcal{S}_h^{k,n}(\varphi_j)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} + \Big\||x|^{\frac{p}{2}} \mathcal{S}_h^{k,n}(\varphi_j)\Big\|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \right) \\ &\geq \frac{1}{2} \left(\frac{3}{||h||^2_{L^2_{k,n}(\mathbb{R})} M(k,n,p) 2^{\frac{12kn+n+12}{n}}} \right)^{\frac{np}{2(2k-1)n+4}} \big(\mathrm{Card}(\mathcal{E})\big)^{1+\frac{np}{2(2k-1)n+4}}. \end{aligned} \tag{7.14}$$

Proof. The result is an immediate consequence of Theorem 7.1 together with the fact that

$$||(x,\nu)||^p \leqslant 2^p(|\nu|^p + |x|^p).$$

□

The dispersion inequality (7.14) implies that there is no infinite sequence $(\varphi_j)_{j\in\mathcal{E}}$ in $L^2_{k,n}(\mathbb{R})$ such that both sequences

$$\Big|\Big|\, |\nu|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)} \quad \text{and} \quad \Big|\Big|\, |x|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}$$

are bounded. More precisely:

Corollary 7.3. *Let $p>0$ and let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. For every $\mathcal{E}\subset\mathbb{N}$, we have*

$$\sup_{j\in\mathcal{E}}\left(\Big|\Big|\, |\nu|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)},\Big|\Big|\, |x|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right)$$
$$\geq \frac{1}{4}\left(\frac{3}{||h||^2_{L^2_{k,n}(\mathbb{R})}M(k,n,p)2^{\frac{12kn+n+12}{n}}}\right)^{\frac{np}{2(2k-1)n+4}}\big(\mathrm{Card}(\mathcal{E})\big)^{\frac{np}{2(2k-1)n+4}}. \quad (7.15)$$

In particular,

$$\sup_{j\in\mathbb{N}}\left(\Big|\Big|\, |\nu|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}+\Big|\Big|\, |x|^{\frac{p}{2}}\mathcal{S}^{k,n}_h(\varphi_j)\Big|\Big|^2_{L^2_{\mu_{k,n}}(\mathbb{R}^2)}\right)=\infty.$$

Theorem 7.2 (Shapiro's Umbrella theorem for $\mathcal{S}^{k,n}_h$)**.** *Let $\mathcal{E}\subset\mathbb{N}$ be a nonempty subset and $(\varphi_j)_{j\in\mathcal{E}}$ be an orthonormal sequence in $L^2_{k,n}(\mathbb{R})$. If there is a positive function $g\in L^2_{\mu_{k,n}}(\mathbb{R}^2)$ such that*

$$|\mathcal{S}^{k,n}_h(\varphi_j)(x,\nu)|\leqslant g(x,\nu)$$

for every $j\in\mathcal{E}$ and for almost every $(x,\nu)\in\mathbb{R}^2$, then $\mathcal{E}$ is finite.

Proof. Following the idea of Malinnikova [20], for every positive real number $\varepsilon<1$, there is a subset $\Delta_{g,\varepsilon}\subset\mathbb{R}^2$ such that

$$\mu_{k,n}(\Delta_{g,\varepsilon})=\inf\left\{\mu_{k,n}(U):\int_{U^c}|g(x,\nu)|^2\,d\mu_{k,n}(x,\nu)\leqslant\varepsilon^2\right\},$$

and

$$\int_{\Delta^c_{g,\varepsilon}}|g(x,\nu)|^2\,d\mu_{k,n}(x,\nu)=\varepsilon^2.$$

Hence, according to the hypothesis, for every $j\in\mathcal{E}$ we have

$$\int_{\Delta^c_{g,\varepsilon}}\left|\mathcal{S}^{k,n}_h(\varphi_j)(x,\nu)\right|^2 d\mu_{k,n}(x,\nu)\leqslant\varepsilon^2,$$

and by Proposition 7.1, we get $\mathrm{Card}(\mathcal{E})(1-\varepsilon)\leqslant\mu_{k,n}(\Delta_{g,\varepsilon})$. □

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