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TIME-FREQUENCY ANALYSIS ASSOCIATED WITH THE DEFORMED WAVELET TRANSFORM

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Abstract. In this paper, we introduce and we study the deformed wavelet transform on $\mathbb{R}^d$. We investigate for this transform some problems of the reproducing kernels theory. In particular we study some time-frequency concentration problems and we give some applications of the Tikhonov regularization on the generalized Sobolev spaces.

1. INTRODUCTION

We consider the differential-difference operators $T_j,\ j = 1, 2, ..., d$, associated with a root system $\mathscr{R}$ and a multiplicity function k, introduced by Dunkl in [6], and called the Dunkl operators in the literature.

The Dunkl theory is based on the Dunkl kernel $K(\lambda, .), \lambda \in \mathbb{C}^d$, which is the unique analytic solution of the system

$$T_j u(x) = \lambda_j u(x), \quad j = 1, 2, ..., d,$$

satisfying the normalizing condition $u(0) = 1$.

With the kernel $K(\lambda, .)$, Dunkl have defined in [7] the Dunkl transform $\mathscr{F}_D$. For a family of weighted functions, ω_k, invariant under a finite reflection group W, Dunkl transform is an extension of the Fourier transform that defines an isometry of $L^2(\mathbb{R}^d, d\gamma_k(x))$ onto itself. The basic properties of the Dunkl transforms have been studied by several authors, see [4, 6, 7, 31] and the references therein.

Very recently, many authors have been investigating the behavior of the Dunkl transform to several problems already studied for the Fourier transform; for instance, Babenko inequality [2], uncertainty principles [9, 12], real Paley-Wiener theorems [15], Dunkl Gabor transform [16, 19], Dunkl wave equation [17], Dunkl Schrödinger equation [18, 20], Dunkl wavelet transform [21, 32], Dunkl-Stockwell transform [22], heat equation [25], Riesz transform [30] and so on.

Despite of the fact that the Fourier transforms have played a significant role in the representation of transient signals by transforming them into the frequency domains, however, they are incompatible for analyzing non-transient signals due to the global kernel. As such, many new integral transforms have been introduced to analyze the non-stationary and multi-component signals in the joint time-frequency domain. The primary ramification in this direction appeared in

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the form of short-time Fourier transform (STFT) (Gabor transform) [8], which performs the localized signal analysis by segmenting the non-stationary signal via rigid window functions. Although the STFT has rectified many limitations of the classical Fourier transform, however, it suffers from the undesirable trade-off between the time-concentration and the frequency concentration due to the rigidity of the window function. Such limitations were addressed by the classical wavelet transform which relies on time-scale adaptable window functions, known as wavelets.

The wavelet transform is a multiscale integral transform, which serves as the pedestal for non-stationary signal processing. Intuitively, the wavelet transform breakdowns a signal into components determined by the translations and dilations of a single function known as the mother wavelet. The wavelet transform has demonstrated to be of great significance in capturing the local properties of non-transient signals by using these local components, and has paved the way for a number of fields including signal and image processing, sampling theory, medicine, turbulence and quantum mechanics [3]. For any $f \in L^2(\mathbb{R}^d)$, the continuous wavelet transform is denoted by $\mathscr{W}_\psi(f)(b,x)$ and is defined as

$$\mathscr{W}_\psi(f)(b,x) = \frac{1}{b^{\frac{d}{2}}} \int_{\mathbb{R}^d} f(y)\, \overline{\psi\left(\frac{y-x}{b}\right)}\, dy, \quad b \in \mathbb{R}^+,\ x \in \mathbb{R}^d, \tag{1.1}$$

where b and x denote the scaling and translation parameters, respectively. For more about wavelet transforms and their applications, we allude to [33, 34].

The purpose of this document is twofold. On one hand, we want to introduce a generalized wavelet transform in the setting of the Dunkl transform and to study its harmonic analysis. Keeping in view the fact that the reproducing kernel theory for the new Dunkl wavelet transforms is yet to be investigated exclusively, our second endeavour is to investigate some problems of the reproducing kernel theory associated with this transform.

The remaining part of the paper is organized as follows. In §2 we recall the main results about the harmonic analysis associated with the Dunkl operators. Next, in §3 we introduce and we study the Dunkl wavelet transform. More precisely the Parseval, Plancherel's and Lieb's formulas are established. In §4, we establish the Heisenberg, Benedicks and Donoho-Stark's type uncertainty principles for the Dunkl wavelet transform. In §5, we study the eigenvalues and eigenfunctions of the time-frequency localization operator. Besides, we also study the scalogram associated with the Dunkl wavelet transform. §6 is devoted to study the Shapiro uncertainty principle for the Dunkl wavelet transform. Finally, in the last section we introduce the generalized Sobolev spaces $W_k^s(\mathbb{R}^d)$ associated with the Dunkl wavelet transform. Afterwards, we give some applications of the general theory of reproducing kernels to the Tikhonov regularization, which gives the best approximation of the Dunkl wavelet transform on these generalized Sobolev spaces.

2. PRELIMINARIES

This section gives an introduction to the Dunkl theory. Main references are [4, 6, 7, 24, 26, 29, 31].

2.1. The Dunkl operators. We consider $\mathbb{R}^d$ with the Euclidean scalar product $\langle , \rangle$ for which the basis $\{e_i,\ i = 1, ..., d\}$ is orthogonal and $||x|| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, let σ_α be the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$ orthogonal to α, i.e.

$$\sigma_\alpha(x) = x - 2\frac{\langle \alpha, x\rangle}{||\alpha||^2}\alpha. \tag{2.1}$$

A finite set $\mathscr{R} \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $\mathscr{R} \bigcap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha(\mathscr{R}) = \mathscr{R}$ for all $\alpha \in \mathscr{R}$. For a given root system $\mathscr{R}$ the reflections $\sigma_\alpha, \alpha \in \mathscr{R}$, generate a finite group $W \subset O(d)$, called the reflection group associated with $\mathscr{R}$.

We fix a positive root system $\mathscr{R}_+ = \Big\{\alpha \in \mathscr{R}\ : \langle \alpha, \beta\rangle > 0\Big\}$ for some β in $\mathbb{R}^d\backslash \bigcup_{\alpha\in\mathscr{R}} H_\alpha$. We will assume that $\langle \alpha, \alpha\rangle = 2$ for all $\alpha \in \mathscr{R}_+$.

A function $k : \mathscr{R} \longrightarrow [0, \infty)$ is called a multiplicity function if it is invariant under the action of the associated reflection group W. For abbreviation, we introduce the index

$$\gamma = \gamma(k) = \sum_{\alpha\in\mathscr{R}_+} k(\alpha). \tag{2.2}$$

Moreover, let ω_k denote the weight function

$$\omega_k(x) = \prod_{\alpha\in\mathscr{R}_+} |\langle \alpha, x\rangle|^{2k(\alpha)}, \tag{2.3}$$

which is W-invariant and homogeneous of degree 2γ.

When $W = \mathbb{Z}_2^d$, the weight function ω_k is given by

$$\omega_k(x) = \prod_{j=1}^{d} |x_j|^{2\alpha_j},\ \alpha_j \geq 0,\ 1 \leq j \leq d.$$

We introduce the Mehta-type constant

$$c_k = \int_{\mathbb{R}^d} e^{-\frac{||x||^2}{2}} \omega_k(x)\, dx. \tag{2.4}$$

In the following we denote by

$C^p(\mathbb{R}^d)$ the space of functions of class C^p on $\mathbb{R}^d$.
$\mathscr{E}(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$.
$\mathscr{S}(\mathbb{R}^d)$ the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
$D(\mathbb{R}^d)$ the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.
$\mathscr{S}'(\mathbb{R}^d)$ the topological dual of the Schwartz space $\mathscr{S}(\mathbb{R}^d)$.

The Dunkl operators $T_j,\ j = 1, ..., d$, on $\mathbb{R}^d$ associated with the finite reflection group W and multiplicity function k are given for f in $C^1(\mathbb{R}^d)$ and $x \in \mathbb{R}^d_{reg} = \mathbb{R}^d\backslash \cup_{\alpha\in\mathscr{R}} H_\alpha$, by

$$T_j f(x) := \frac{\partial f}{\partial x_j}(x) + \sum_{\alpha\in\mathscr{R}_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle}, \quad f \in C^1(\mathbb{R}^d), \tag{2.5}$$

where $\alpha_j = \langle \alpha, e_j \rangle$.

The Dunkl operators form a commutative system of differential-difference operators.

We define the Dunkl-Laplacian operator $\triangle_k$ on $\mathbb{R}^d$ for f in $C^2(\mathbb{R}^d)$ and $x \in \mathbb{R}^d_{reg}$, by

$$\triangle_k f(x) := \sum_{j=1}^{d} T_j^2 f(x) = \triangle f(x) + 2 \sum_{\alpha \in \mathscr{R}^+} k(\alpha) \Big(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \Big),$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and the gradient operators on $\mathbb{R}^d$ respectively.

For $y \in \mathbb{R}^d$, the system

$$\left\{ \begin{array}{lcl} T_j u(x,y) & = & y_j u(x,y), \quad j = 1, ..., d, \\ u(0,y) & = & 1, \end{array} \right. \tag{2.6}$$

admits a unique analytic solution on $\mathbb{R}^d$, which will be denoted by $K(x,y)$ and called Dunkl kernel. This kernel has a unique holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$.

The Dunkl kernel possesses the following properties:

i) For $z, t \in \mathbb{C}^d$, we have $K(z,t) = K(t,z); K(z,0) = 1$ and $K(\lambda z, t) = K(z, \lambda t)$ for all $\lambda \in \mathbb{C}$.

ii) For all $\nu \in \mathbb{N}^d, x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ we have

$$|D_z^\nu K(x,z)| \leq ||x||^{|\nu|} \exp(||x|| \, ||\mathrm{Re} z||), \tag{2.7}$$

with

$$D_z^\nu = \frac{\partial^{|\nu|}}{\partial z_1^{\nu_1} \cdots \partial z_d^{\nu_d}} \quad \text{and} \quad |\nu| = \nu_1 + \cdots + \nu_d.$$

In particular for all $x, y \in \mathbb{R}^d$:

$$|K(ix,y)| \leq 1.$$

iii) The function $K(x,z)$ admits for all $x \in \mathbb{R}^d$ and $z \in \mathbb{C}^d$ the following Laplace type integral representation

$$K(x,z) = \int_{\mathbb{R}^d} e^{\langle y,z \rangle} d\nu_x(y), \tag{2.8}$$

where ν_x is a positive probability measure on $\mathbb{R}^d$, with support in the closed ball $B(0, ||x||)$ of center 0 and radius $||x||$. (See [25]).

2.2. The Dunkl transform.

Notation. We denote by $L^p_k(\mathbb{R}^d)$ the space of measurable functions on $\mathbb{R}^d$ such that

$$\begin{array}{lcl} ||f||_{L^p_k(\mathbb{R}^d)} & := & \Big(\int_{\mathbb{R}^d} |f(x)|^p \, d\gamma_k(x) \Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty, \\ ||f||_{L^\infty_k(\mathbb{R}^d)} & := & \operatorname{ess} \sup_{x \in \mathbb{R}^d} |f(x)| < \infty, \end{array}$$

where

$$d\gamma_k(x) := \frac{1}{c_k} \omega_k(x) dx.$$

For $p=2$, we provide this space with the scalar product

$$\langle f,g\rangle_{L^2_k(\mathbb{R}^d)} := \int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x).$$

If $\mathscr{F}$ is a space of a $\mathbb{C}$-valued functions on $\mathbb{R}^d$, denote by

$$\mathscr{F}_{rad} := \Big\{ f\in\mathscr{F} : f\circ A = f \text{ for all } A\in O(d,\mathbb{R}) \Big\}$$

the subspace of those $f\in\mathscr{F}$ which are radial. For $f\in\mathscr{F}_{rad}$ there exists a unique function $F:\mathbb{R}_+\to\mathbb{C}$ such that $f(x)=F(||x||)$ for all $x\in\mathbb{R}^d$.

Remark 2.1. *By using the homogeneity of ω_k it is shown in [24] that for a radial function $f\in L^1_k(\mathbb{R}^d)$ the function F defined on $[0,\infty)$ by $f(x)=F(||x||)$, for all $x\in\mathbb{R}^d$ is integrable with respect to the measure $r^{2\gamma+d-1}dr$. More precisely,*

$$\int_{\mathbb{R}^d} f(x)d\gamma_k(x) = d_k \int_0^\infty F(r)r^{2\gamma+d-1}dr, \tag{2.9}$$

where

$$d_k := \frac{1}{2^{\gamma+\frac{d}{2}-1}\Gamma(\gamma+\frac{d}{2})}. \tag{2.10}$$

The Dunkl transform of a function f in $L^1_k(\mathbb{R}^d)$ is given by

$$\mathscr{F}_D(f)(y) = \int_{\mathbb{R}^d} f(x)K(-ix,y)d\gamma_k(x), \quad \text{for all } y\in\mathbb{R}^d. \tag{2.11}$$

In the following we give some properties of this transform (cf. [4, 7]).

i) For f in $L^1_k(\mathbb{R}^d)$ we have

$$||\mathscr{F}_D(f)||_{L^\infty_k(\mathbb{R}^d)} \le ||f||_{L^1_k(\mathbb{R}^d)}. \tag{2.12}$$

ii) Inversion formula: Let f be a function in $L^1_k(\mathbb{R}^d)$, such that $\mathscr{F}_D(f)\in L^1_k(\mathbb{R}^d)$. Then

$$\mathscr{F}_D^{-1}(f)(x) = \mathscr{F}_D(f)(-x), \quad a.e.\ x\in\mathbb{R}^d. \tag{2.13}$$

Proposition 2.1. *The Dunkl transform $\mathscr{F}_D$ is a topological isomorphism from $\mathscr{S}(\mathbb{R}^d)$ onto itself. If we put for f in $\mathscr{S}(\mathbb{R}^d)$*

$$\overline{\mathscr{F}_D}(f)(y) = \mathscr{F}_D(f)(-y), \quad y\in\mathbb{R}^d, \tag{2.14}$$

we have

$$\mathscr{F}_D\overline{\mathscr{F}_D} = \overline{\mathscr{F}_D}\mathscr{F}_D = Id.$$

Proposition 2.2. i*) Plancherel's formula for $\mathscr{F}_D$.*
For all f in $\mathscr{S}(\mathbb{R}^d)$ we have

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 d\gamma_k(\xi). \tag{2.15}$$

ii*) Plancherel's theorem for $\mathscr{F}_D$.*
The Dunkl transform $f\mapsto\mathscr{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L^2_k(\mathbb{R}^d)$.

iii*) Parseval's formula for* $\mathscr{F}_D$.
For all f, g *in* $\mathscr{S}(\mathbb{R}^d)$ *we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x) = \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)}d\gamma_k(\xi). \tag{2.16}$$

Definition 2.1. *Let* U, V *be two measurable subsets of* $\mathbb{R}^d$. *Then:*

(1) We say that the pair (U, V) *is weakly annihilating, if supp* $f \subset U$ *and supp* $\mathscr{F}_D(f) \subset V$ *implies* $f = 0$.

(2) We say that the pair (U, V) *is strongly annihilating, if there exists a positive constant* $C := C_k(U, V)$ *such that for every function* f *in* $L^2_k(\mathbb{R}^d)$,

$$C\big(\|\mathscr{F}_D(f)\|^2_{L^2_k(V^c)} + ||f||^2_{L^2_k(U^c)}\big) \geq \|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{2.17}$$

Here $A^c := \mathbb{R}^d \backslash A$ *is the complement of* A. *The constant* $C_k(U, V)$ *will be called the annihilation constant of* (U, V).

Now, we recall the following Benedicks-type uncertainty principle for the Dunkl transform proved by Ghobber and Jaming in [10].

Proposition 2.3. *Let* U, V *be two measurable subsets of* $\mathbb{R}^d$ *with*

$$\gamma_k(U) := \int_U d\gamma_k(x) < \infty \quad \text{and} \quad \gamma_k(V) := \int_V d\gamma_k(x) < \infty.$$

Then the pair (U, V) *is a strongly annihilating pair.*

Definition 2.2. *([24]) Let* $x \in \mathbb{R}^d$. *The Dunkl translation operator* $f \mapsto \tau_x f$ *is defined on* $L^2_k(\mathbb{R}^d)$ *by*

$$\mathscr{F}_D(\tau_x f) = K(ix, .)\mathscr{F}_D(f). \tag{2.18}$$

It is useful to have a class of functions in which (2.18) holds pointwise. One such class is given by the generalized Wigner space $\mathscr{W}_k(\mathbb{R}^d)$ given by

$$\mathscr{W}_k(\mathbb{R}^d) := \Big\{f \in L^1_k(\mathbb{R}^d) : \mathscr{F}_D(f) \in L^1_k(\mathbb{R}^d)\Big\}.$$

Proposition 2.4. *([24, 29]) i) Let* $x \in \mathbb{R}^d$. *For all* f *in* $L^2_k(\mathbb{R}^d)$, *we have*

$$\|\tau_x f\|_{L^2_k(\mathbb{R}^d)} \leq \|f\|_{L^2_k(\mathbb{R}^d)}.$$

ii) For all f *in* $\mathscr{W}_k(\mathbb{R}^d)$ *and* $x \in \mathbb{R}^d$, *we have for every* $y \in \mathbb{R}^d$

$$\tau_x f(y) = \int_{\mathbb{R}^d} K(ix, \xi)K(iy, \xi)\mathscr{F}_D(f)(\xi)d\gamma_k(\xi).$$

iii) For all f *in* $\mathscr{W}_k(\mathbb{R}^d)$, *we have for* $x, y \in \mathbb{R}^d$

$$\tau_x f(y) = \tau_y(f)(x). \tag{2.19}$$

At the moment an explicit formula for the Dunkl translation operators is known only in the following two cases.

$\underline{1^{st}\ \text{case}}$: $d=1$ and $W=\mathbb{Z}_2$.
For all $f\in C(\mathbb{R})$ we have for all $x\in\mathbb{R}$,

$$\begin{aligned}\tau_y f(x) &= \tfrac{1}{2}\int_{-1}^{1} f(\sqrt{x^2+y^2-2xyt})(1+\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\Phi_k(t)d\gamma_k(t)\\ &+ \tfrac{1}{2}\int_{-1}^{1} f(-\sqrt{x^2+y^2-2xyt})(1-\frac{x-y}{\sqrt{x^2+y^2-2xyt}})\Phi_k(t)d\gamma_k(t),\end{aligned}$$

where

$$\Phi_k(t)=\frac{\Gamma(k+\frac{1}{2})}{\sqrt{\pi}\Gamma(k)}(1+t)(1-t^2)^{k-1},$$

from which also follows a formula of $\tau_y f$ in the case of $W=\mathbb{Z}_2^d$. The explicit formula implies the L^p-boundedness of $\tau_y f$. More precisely, we have.

Proposition 2.5. *([26, 29]) Let $W=\mathbb{Z}_2^d$. For all $f\in L_k^p(\mathbb{R}^d)$, $1\le p\le\infty$, we have*

$$||\tau_y f||_{L_k^p(\mathbb{R}^d)}\le 2^{d|\frac{1}{p}-\frac{1}{2}|}||f||_{L_k^p(\mathbb{R}^d)}. \tag{2.20}$$

$\underline{2^{nd}\ \text{case}}$: For all radial function f in $\mathscr{W}_k(\mathbb{R}^d)$ we have

$$\forall\, x\in\mathbb{R}^d,\ \tau_y f(x)=\int_{B_d(0,||x||)} f_0(\sqrt{||x||^2+||y||^2+2\langle x,z\rangle})d\nu_x(z),$$

with f_0 the function on $[0,\infty)$ given by $f(x)=f_0(||x||)$ and ν_x the measure given by (2.8).

Several essential properties of $\tau_y f$ is established for f being radial functions. This is collected in the following proposition (see [29]). Let $L^p_{k,rad}(\mathbb{R}^d)$ stand for the subspace of radial functions in $L_k^p(\mathbb{R}^d)$.

Proposition 2.6. *(i) Let f be in $L^1_{k,rad}(\mathbb{R}^d)$ and nonnegative. Then we have*

$$\forall\, y\in\mathbb{R}^d,\quad \tau_y f\ge 0,\quad \tau_y f\in L_k^1(\mathbb{R}^d)$$

and

$$\int_{\mathbb{R}^d}\tau_y f(x)d\gamma_k(x)=\int_{\mathbb{R}^d} f(x)d\gamma_k(x). \tag{2.21}$$

(ii) Let f be in $L^p_{k,rad}(\mathbb{R}^d)$, $1\le p\le\infty$, then we have

$$\forall\, y\in\mathbb{R}^d,\quad ||\tau_y f||_{L_k^p(\mathbb{R}^d)}\le ||f||_{L_k^p(\mathbb{R}^d)}. \tag{2.22}$$

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (see [29, 31]).

Definition 2.3. *For f, g in $D(\mathbb{R}^d)$, we define the Dunkl convolution product by*

$$\forall\, x\in\mathbb{R}^d,\quad f*_D g(x)=\int_{\mathbb{R}^d}\tau_x f(-y)g(y)d\gamma_k(y). \tag{2.23}$$

This convolution is commutative, associative and satisfies the following properties (see [29, 31]).

Proposition 2.7. *i) Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is a radial function in $L^p_k(\mathbb{R}^d)$ and g an element of $L^q_k(\mathbb{R}^d)$, then $f *_D g$ belongs to $L^r_k(\mathbb{R}^d)$ and we have*

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)}. \tag{2.24}$$

ii) Let $W = \mathbb{Z}_2^d$. For all f in $L^p_k(\mathbb{R}^d)$ and g an element of $L^q_k(\mathbb{R}^d)$, the function $f *_D g$ belongs to $L^r_k(\mathbb{R}^d)$ with $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$ and we have

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq 2^{d|\frac{1}{p} - \frac{1}{2}|} \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)}. \tag{2.25}$$

Proposition 2.8. *(i) Let f and g be in $L^2_k(\mathbb{R}^d)$. Then, the function $f *_D g$ belongs to $L^2_k(\mathbb{R}^d)$, if and only if the function $\mathscr{F}_D(f)\mathscr{F}_D(g)$ is in $L^2_k(\mathbb{R}^d)$, and we have*

$$\mathscr{F}_D(f *_D g) = \mathscr{F}_D(f)\mathscr{F}_D(g),$$

in the L^2-case.

(ii) Let f and g be in $L^2_k(\mathbb{R}^d)$. Then, we have

$$\int_{\mathbb{R}^d} |f *_D g(x)|^2 d\gamma_k(x) = \int_{\mathbb{R}^d} |\mathscr{F}_D(f)(\xi)|^2 |\mathscr{F}_D(g)(\xi)|^2 d\gamma_k(\xi), \tag{2.26}$$

both members are finite or infinite.

In the rest of this paper we assume that $W = \mathbb{Z}_2^d$.

3. DEFORMED WAVELET TRANSFORM

Let $a := (a_1, ..., a_d) \in \mathbb{R}^d$. The dilation operator Δ_a of a measurable function h, is defined by

$$\forall\, x \in \mathbb{R}^d,\ \Delta_a h(x) := |a_1|^{\frac{2\alpha_1+1}{2}} ... |a_d|^{\frac{2\alpha_d+1}{2}} h(a_1 x_1, ..., a_d x_d). \tag{3.1}$$

By simple calculations we prove that these operators satisfy the following properties.

Proposition 3.1. (i) *For all a, b in $\mathbb{R}^d$, we have*

$$\Delta_a \Delta_b = \Delta_{(a_1 b_1, ..., a_d b_d)}. \tag{3.2}$$

(ii) *Let $a \in \mathbb{R}^d \backslash \{0_{\mathbb{R}^d}\}$. For all h in $L^2_k(\mathbb{R}^d)$, the function $\Delta_a h$ belongs to $L^2_k(\mathbb{R}^d)$ and we have*

$$||\Delta_a h||_{L^2_k(\mathbb{R}^d)} = ||h||_{L^2_k(\mathbb{R}^d)}, \tag{3.3}$$

and

$$\mathscr{F}_D(\Delta_a h)(y) = |a_1|^{-\frac{2\alpha_1+1}{2}} ... |a_d|^{-\frac{2\alpha_d+1}{2}} \mathscr{F}_D(h)(\frac{y_1}{a_1}, ..., \frac{y_d}{a_d}), \quad y \in \mathbb{R}^d. \tag{3.4}$$

(iii) *Let $a \in \mathbb{R}^d \backslash \{0_{\mathbb{R}^d}\}$. For all h, g in $L^2_k(\mathbb{R}^d)$, we have*

$$\langle \Delta_a h, g \rangle_{L^2_k(\mathbb{R}^d)} = \langle h, \Delta_{(\frac{1}{a_1}, ..., \frac{1}{a_d})} g \rangle_{L^2_k(\mathbb{R}^d)}. \tag{3.5}$$

(iv) *Let $a \in \mathbb{R}^d \backslash \{0_{\mathbb{R}^d}\}$ and $x \in \mathbb{R}^d$. We have*

$$\Delta_a \tau_x = \tau_{(\frac{x_1}{a_1}, ..., \frac{x_d}{a_d})} \Delta_a. \tag{3.6}$$

Definition 3.1. *A Dunkl wavelet on $\mathbb{R}^d$ is a measurable function h on $\mathbb{R}^d$ satisfying for almost all $\xi \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$, the condition*

$$0 < C_h := \int_{\mathbb{R}^d} |\mathscr{F}_D(\Delta_a h)(\xi)|^2 d\gamma_k(a) < \infty. \tag{3.7}$$

Proposition 3.2. *Let h be a Dunkl wavelet on $\mathbb{R}^d$, then we have*

$$C_h := \int_{\mathbb{R}^d} |\mathscr{F}_D(\Delta_a h(\xi)|^2 d\gamma_k(a) = \frac{1}{c_k}\int_{\mathbb{R}^d} |\mathscr{F}_D(h)(\frac{\xi_1}{a_1},...,\frac{\xi_d}{a_d})|^2 \frac{da_1}{|a_1|}...\frac{da_d}{|a_d|}.$$

Proof. Let $a \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$. Using the relations (3.4) and (3.1) we deduce that

$$|\mathscr{F}_D(\Delta_a h)(\xi)|^2 \quad = \quad \frac{1}{|a_1|^{2\alpha_1+1}...|a_d|^{2\alpha_d+1}}|\mathscr{F}_D(h)(\frac{\xi_1}{a_1},...,\frac{\xi_d}{a_d})|^2. \tag{3.8}$$

When $W = \mathbb{Z}_2^d$, we have $d\gamma_k(a) = \frac{1}{c_k}\prod_{j=1}^d |a_j|^{2\alpha_j} da_j$. Then using (3.7) we get

$$\begin{aligned} C_h : &= \int_{\mathbb{R}^d} |\mathscr{F}_D(\Delta_a h)(\xi)|^2 d\gamma_k(a) \\ &= \int_{\mathbb{R}^d} |\mathscr{F}_D(h)(\frac{\xi_1}{a_1},...,\frac{\xi_d}{a_d})|^2 \frac{d\gamma_k(a)}{|a_1|^{2\alpha_1+1}...|a_d|^{2\alpha_d+1}} \\ &= \frac{1}{c_k}\int_{\mathbb{R}^d} |\mathscr{F}_D(h)(\frac{\xi_1}{a_1},...,\frac{\xi_d}{a_d})|^2 \frac{da_1}{|a_1|}...\frac{da_d}{|a_d|}. \end{aligned}$$

Thus we obtain the desired result. □

Example 3.1. *The function α_t, $t > 0$, defined on $\mathbb{R}^d$ by*

$$\alpha_t(x) = \frac{1}{(2t)^{\gamma+\frac{d}{2}}} e^{-\frac{||x||^2}{4t}}, \tag{3.9}$$

satisfies

$$\forall\, y \in \mathbb{R}^d,\ \mathscr{F}_D(\alpha_t)(y) = e^{-t||y||^2}. \tag{3.10}$$

The function $h(x) = -\dfrac{d}{dt}\alpha_t(x)$ is a Dunkl wavelet on $\mathbb{R}^d$ in $\mathscr{S}(\mathbb{R}^d)$.

Let $a \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$ and h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. We consider the family $h_{a,x}$, $x \in \mathbb{R}^d$, of functions on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ defined by

$$h_{a,x}(y) := \tau_x(\Delta_a h)(y),\ y \in \mathbb{R}^d, \tag{3.11}$$

where τ_x, $x \in \mathbb{R}^d$, are the Dunkl translation operators given by (2.18).

We note that we have

$$\forall\, a \in \mathbb{R}^d, \forall\, x \in \mathbb{R}^d, \quad ||h_{a,x}||_{L^2_k(\mathbb{R}^d)} \leq ||h||_{L^2_k(\mathbb{R}^d)}. \tag{3.12}$$

For $1 \leqslant p \leqslant \infty$, let $L^p_{\mu_k}(\mathbb{R}^{2d})$, $p \in [1,\infty]$, be the space of measurable functions f on $\mathbb{R}^{2d}$ such that

$$\begin{aligned} \|f\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} &:= \left(\int_{\mathbb{R}^{2d}} |f(a,x)|^p d\mu_k(a,x)\right)^{\frac{1}{p}} < \infty, \quad 1 \leq p < \infty, \\ \|f\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} &:= \underset{(a,x)\in\mathbb{R}^{2d}}{ess\sup}\ |f(a,x)| < \infty, \end{aligned}$$

where the measure μ_k is defined by

$$\forall\, (a,x) \in \mathbb{R}^{2d}, \quad d\mu_k(a,x) = d\gamma_k(x)d\gamma_k(a).$$

Definition 3.2. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. The Dunkl continuous wavelet transform $\mathscr{N}_h^D$ on $\mathbb{R}^d$ is defined for regular functions f on $\mathbb{R}^d$ by*

$$\mathscr{N}_h^D(f)(a,x) = \int_{\mathbb{R}^d} f(y)\overline{h_{a,x}(y)}d\gamma_k(y) = \langle f, \tau_x\Delta_a h\rangle_{L^2_k(\mathbb{R}^d)},\ a,x\in\mathbb{R}^d. \quad (3.13)$$

This transform can also be written in the form

$$\mathscr{N}_h^D(f)(a,x) = \breve{f} *_D \overline{\Delta_a h}(x), \quad (3.14)$$

where $\breve{f}(y) := f(-y)$, and $*_D$ is the Dunkl convolution product given by (2.23).

Remark 3.1. *(i) Let h be a Dunkl wavelet in $L^2_k(\mathbb{R}^d)$. Using relation (3.13), Cauchy-Schwarz's inequality and relation (3.12) we get, for all f in $L^2_k(\mathbb{R}^d)$*

$$\|\mathscr{N}_h^D(f)\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \le \|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \quad (3.15)$$

(ii) Using Proposition 3.1 and by a standard computation it is easy to see that, for every f and h in $L^2_k(\mathbb{R}^d)$, for all $\lambda > 0$ and for all $(a,x)\in\mathbb{R}^{2d}$, we have

$$\mathscr{N}_h^D(f_\lambda)(a,x) = \mathscr{N}_h^D(f)(\lambda a, \frac{x}{\lambda}), \quad (3.16)$$

where

$$\forall\, t>0,\ \forall\, x\in\mathbb{R}^d,\ g_t(x) := \frac{1}{t^{\frac{2\gamma+d}{2}}}g(\frac{x}{t}).$$

By standard calculations we prove that:

Lemma 3.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$, then for any $f\in L^2_k(\mathbb{R}^d)$, we have*

$$\mathscr{F}_D\Big(\mathscr{N}_h^D(f)(a,.)\Big)(\xi) = \mathscr{F}_D(\overline{\Delta_a h})(\xi)\mathscr{F}_D(f)(-\xi). \quad (3.17)$$

Theorem 3.1. *(Parseval's formula for $\mathscr{N}_h^D$). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ and f,g in $L^2_k(\mathbb{R}^d)$. Then, we have*

$$\int_{\mathbb{R}^d} f(x)\overline{g(x)}d\gamma_k(x) = \frac{1}{C_h}\int_{\mathbb{R}^{2d}} \mathscr{N}_h^D(f)(a,x)\overline{\mathscr{N}_h^D(g)(a,x)}d\mu_k(a,x). \quad (3.18)$$

Proof. Using Fubini's Theorem, relation (3.14) and Parseval's formula (2.16), we get

$$\begin{aligned}
&\int_{\mathbb{R}^{2d}} \mathscr{N}_h^D(f)(a,x)\overline{\mathscr{N}_h^D(g)(a,x)}d\mu_k(a,x)\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \big(\breve{f} *_D \overline{\Delta_a h}(x)\big)\overline{\breve{g} *_D \overline{\Delta_a h}(x)}d\mu_k(a,x)\\
&= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{F}_D(\breve{f})(\xi)\overline{\mathscr{F}_D(\breve{g})(\xi)}|\mathscr{F}_D(\overline{\Delta_a h})(\xi)|^2 d\gamma_k(\xi)d\gamma_k(a)\\
&= \int_{\mathbb{R}^d} \mathscr{F}_D(f)(-\xi)\overline{\mathscr{F}_D(g)(-\xi)}\Big(\int_{\mathbb{R}^d}|\mathscr{F}_D(\Delta_a h)(\xi)|^2 d\gamma_k(a)\Big)d\gamma_k(\xi)\\
&= \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)}\Big(\int_{\mathbb{R}^d}|\mathscr{F}_D(\Delta_a h)(-\xi)|^2 d\gamma_k(a)\Big)d\gamma_k(\xi).
\end{aligned}$$

As h is a Dunkl wavelet, (3.7) gives that

$$\int_{\mathbb{R}^d} |\mathscr{F}_D(\Delta_a h)(-\xi)|^2 d\gamma_k(a) \quad = \quad C_h.$$

Thus we obtain

$$\int_{\mathbb{R}^{2d}} \mathscr{N}_h^D(f)(a,x)\overline{\mathscr{N}_h^D(g)(a,x)}d\mu_k(a,x) \quad = \quad C_h \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)}d\gamma_k(\xi).$$

Finally using Proposition 2.2 iii) we obtain the result. □

Corollary 3.1. *(Plancherel's formula for $\mathscr{N}_h^D$). Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L_k^2(\mathbb{R}^d)$. For all f in $L_k^2(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} |f(x)|^2 d\gamma_k(x) = \frac{1}{C_h}\int_{\mathbb{R}^{2d}} |\mathscr{N}_h^D(f)(a,x)|^2 d\mu_k(a,x). \tag{3.19}$$

By Riesz-Thorin's interpolation theorem we obtain the following:

Proposition 3.3. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L_k^2(\mathbb{R}^d)$, $f \in L_k^2(\mathbb{R}^d)$ and p belongs in $[2,\infty]$. We have*

$$\|\mathscr{N}_h^D(f)\|_{L^p_{\mu_k}(\mathbb{R}^{2d})} \leq (C_h)^{\frac{1}{p}}(\|h\|_{L_k^2(\mathbb{R}^d)})^{\frac{p-2}{p}}\|f\|_{L_k^2(\mathbb{R}^d)}. \tag{3.20}$$

4. TIME-FREQUENCY CONCENTRATION

Proposition 4.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L_k^2(\mathbb{R}^d)$. Then, $\mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$ is a reproducing kernel Hilbert space with kernel function*

$$\mathscr{K}_h(b',x';b,x) := \frac{1}{C_h}\int_{\mathbb{R}^d} h_{b',x'}(y)\overline{h_{b,x}(y)}d\gamma_k(y). \tag{4.1}$$

The kernel $\mathscr{K}_h$ satisfies :

$$\forall\,(b',x'),\,(b,x) \in \mathbb{R}^{2d}, \quad |\mathscr{K}_h(b',x';b,x)| \leq \frac{||h||^2_{L_k^2(\mathbb{R}^d)}}{C_h}. \tag{4.2}$$

Proof. Let f be in $L_k^2(\mathbb{R}^d)$. We have

$$\mathscr{N}_h^D(f)(b,x) = \int_{\mathbb{R}^d} f(y)\overline{h_{b,x}(y)}d\gamma_k(y),\ (b,x) \in \mathbb{R}^{2d}.$$

Using relation (3.18), we obtain

$$\mathscr{N}_h^D(f)(b,x) = \frac{1}{C_h}\int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{N}_h^D(f)(b',x')\overline{\mathscr{N}_h^D(h_{b,x})(b',x')}d\mu_k(b',x').$$

On the other hand, using Proposition 2.8, one can easily see that the function

$$x' \mapsto \frac{1}{C_h}\overline{\mathscr{N}_h^D(h_{b,x})(b',x')} = \frac{1}{C_h}\int_{\mathbb{R}^d} h_{b',x'}(y)\overline{h_{b,x}(y)}d\gamma_k(y)$$

belongs to $L_k^2(\mathbb{R}^d)$, for every $(b,x),(b',x') \in \mathbb{R}^{2d}$. Therefore, the result is obtained. □

In order to prove a concentration result of the Dunkl wavelet continuous transform, we need the following notations:

$P_h : L^2_{\mu_k}(\mathbb{R}^{2d}) \to L^2_{\mu_k}(\mathbb{R}^{2d})$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^{2d})$ onto $\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))$.

$P_U : L^2_{\mu_k}(\mathbb{R}^{2d}) \to L^2_{\mu_k}(\mathbb{R}^{2d})$ the orthogonal projection from $L^2_{\mu_k}(\mathbb{R}^{2d})$ onto the subspace of functions of $L^2_{\mu_k}(\mathbb{R}^{2d})$ onto the subspace of function supported in the subset $U \subset \mathbb{R}^{2d}$. Otherwise, say we can write

$$P_U F = \chi_U F, \quad F \in L^2_{\mu_k}(\mathbb{R}^{2d}),$$

where χ_U denotes the characteristic function of the subset U of $\mathbb{R}^{2d}$.

We put

$$||P_U P_h|| := \sup\Big\{||P_U P_h v||_{L^2_{\mu_k}(\mathbb{R}^{2d})} : \ v \in L^2_{\mu_k}(\mathbb{R}^{2d}), \ ||v||_{L^2_{\mu_k}(\mathbb{R}^{2d})} = 1\Big\}.$$

In the following we will prove the concentration of $\mathscr{N}_h^D(f)$ in small sets.

Proposition 4.2. *If the subset $U \subset \mathbb{R}^{2d}$ verifies*

$$\mu_k(U) := \int_U d\mu_k(b,x) < \frac{C_h}{||h||^2_{L^2_k(\mathbb{R}^d)}}.$$

Then, for all f in $L^2_k(\mathbb{R}^d)$ we have

$$\|\chi_{U^c} \mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq \sqrt{C_h}\sqrt{1 - \frac{||h||^2_{L^2_k(\mathbb{R}^d)}}{C_h}\mu_k(U)}\|f\|_{L^2_k(\mathbb{R}^d)}, \tag{4.3}$$

where χ_{U^c} denotes the characteristic function of the complementary U^c of U.

Proof. From Plancherel's formula (3.19) we have

$$C_h||f||^2_{L^2_k(\mathbb{R}^d)} = \|\mathscr{N}_h^D(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} = \|\mathscr{N}_h^D(f)\|^2_{L^2_{\mu_k}(U)} + \|\mathscr{N}_h^D(f)\|^2_{L^2_{\mu_k}(U^c)}. \tag{4.4}$$

On the other hand from the relation (3.15) we have

$$\int_U |\mathscr{N}_h^D(f)(b,x)|^2 d\mu_k(b,x) \leq \mu_k(U)\|f\|^2_{L^2_k(\mathbb{R}^d)}\|h\|^2_{L^2_k(\mathbb{R}^d)}. \tag{4.5}$$

Thus the relation (4.3) follows immediately from the relations (4.4), (4.5). □

Remark 4.1. *If $\mathscr{N}_h^D(f)$ is supported in U and $\mu_k(U) < \frac{C_h}{||h||^2_{L^2_k(\mathbb{R}^d)}}$, then $f = 0$.*

Proposition 4.3. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$.*

Let $s > 0$. Then the following uncertainty inequalities hold.

(1) *A Heisenberg-type uncertainty inequalities for $\mathscr{N}_h^D$:*
   (i) *There exists a constant $C_1(k,s,h) > 0$ such that, for all f in $L^2_k(\mathbb{R}^d)$, we have*

$$\Big|\Big|\,||(a,x)||^s \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq C_1(k,s,h)\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{4.6}$$

(ii) There exists a constant $C_2(k,s,h) > 0$ such that, for all f in $L^2_k(\mathbb{R}^d)$, we have

$$\Big|\Big|\, ||x||^s \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big|\Big|\, ||a||^s \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq C_2(k,s,h)\|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{4.7}$$

(2) A Faris local uncertainty inequality for $\mathscr{N}_h^D$:
There exists a constant $C_3(k,s,h) > 0$ such that, for all f in $L^2_k(\mathbb{R}^d)$, and every subset $U \subset \mathbb{R}^{2d}$ such that $0 < \mu_k(U) < \infty$, we have

$$||\mathscr{N}_h^D(f)||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leqslant C_3(k,s,h)\sqrt{\mu_k(U)}\Big|\Big|\, ||(a,x)||^s \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.8}$$

Proof. (1) Let $r > 0$ such that $0 < \mu_k(B_{2d}(0,r)) < \frac{C_h}{\|h\|^2_{L^2_k(\mathbb{R}^d)}}$ where $B_{2d}(0,r)$ is the open ball of $\mathbb{R}^{2d}$ defined by

$$B_{2d}(0,r) = \Big\{(a,x) \in \mathbb{R}^{2d} : ||(a,x)|| < r\Big\}.$$

Involving the relation (4.3) with $U = B_{2d}(0,r)$, and by simple calculation we obtain

$$\begin{aligned}\big(C_h - \mu_k(U)\|h\|^2_{L^2_k(\mathbb{R}^d)}\big)||f||^2_{L^2_k(\mathbb{R}^d)} &\leqslant \int_{B_{2d}(0,r)^c} |\mathscr{N}_h^D(f)(a,x)|^2 d\mu_k(a,x)\\ &\leqslant \tfrac{1}{r^{2s}}\Big|\Big|\, ||(a,x)||^s \mathscr{N}_h^D(f)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

Thus we obtain (4.6) with $C_1(k,s,h) := r^s\sqrt{C_h - \mu_k(U)\|h\|^2_{L^2_k(\mathbb{R}^d)}}$.

(ii) By applying the inequality $||(a,x)||^s \leqslant 2^s(||a||^s + ||x||^s)$ in (4.6), we get

$$\Big|\Big|\, ||x||^s \mathscr{N}_h^D(f)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + ||\, ||a||^s \mathscr{N}_h^D(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq \frac{\big(C_1(k,s,h)\big)^2}{2^{2s}}\|f\|^2_{L^2_k(\mathbb{R}^d)}. \tag{4.9}$$

We replace f by f_t, in the relation (4.9), we apply (3.16) and next we make a change of variables in each term, we obtain the following relation:

$$t^{2s}||\, ||x||^s \mathscr{N}_h^D(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + t^{-2s}||\, ||a||^s \mathscr{N}_h^D(f)||^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq \frac{\big(C_1(k,s,h)\big)^2}{2^{2s}}\|f\|^2_{L^2_k(\mathbb{R}^d)}.$$

Then (4.7) follows by minimizing the left hand side of this inequality, with respect $t > 0$.

(2) Using the fact that

$$||\mathscr{N}_h^D(f)||_{L^2_{\mu_k}(U)} \leqslant \sqrt{\mu_k(U)}||\mathscr{N}_h^D(f)||_{L^\infty_{\mu_k}(\mathbb{R}^{2d})},$$

and the fact that

$$||\mathscr{N}_h^D(f)||_{L^\infty_{\mu_k}(\mathbb{R}^{2d})} \leqslant ||h||_{L^2_k(\mathbb{R}^d)}||f||_{L^2_k(\mathbb{R}^d)},$$

then we get

$$||\mathscr{N}_h^D(f)||_{L^2_{\mu_k}(U)} \leqslant \sqrt{\mu_k(U)}||h||_{L^2_k(\mathbb{R}^d)}||f||_{L^2_k(\mathbb{R}^d)}.$$

Finally, we obtain the result from (4.6). □

We shall prove in the following the concentration of $\mathscr{N}_h^D(f)$ in arbitrary sets of finite measures.

Theorem 4.1. *We consider the subset U of $\mathbb{R}^{2d}$ satisfying the following relation*

$$0 < \mu_k(U) < \infty. \tag{4.10}$$

If $P_h(L^2_{\mu_k}(\mathbb{R}^{2d})) \cap P_U(L^2_{\mu_k}(\mathbb{R}^{2d})) = \{0\}$, then, there exists a positive constant $C := C_k(h,U)$ such that for all f in $L^2_k(\mathbb{R}^d)$, we have

$$\|\chi_{U^c}\mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq C\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{4.11}$$

For the proof of this theorem, we need the following Lemma.

Lemma 4.1. *([11]). Let $\mathscr{H}_1$ and $\mathscr{H}_2$ be two closed subspaces of a Hilbert space $\mathscr{H}$ such that $\mathscr{H}_1 \bigcap \mathscr{H}_2 = \{0\}$. Let $P_{\mathscr{H}_1}$ and $P_{\mathscr{H}_2}$ denote the corresponding orthogonal projections, and assume that the product $P_{\mathscr{H}_1}P_{\mathscr{H}_2}$ is a compact operator. Then, there exists a constant $C > 0$ such that for all f in $\mathscr{H}$, we have*

$$||P_{\mathscr{H}_1^{\perp}} f||_{\mathscr{H}} + ||P_{\mathscr{H}_2^{\perp}} f||_{\mathscr{H}} \geq C||f||_{\mathscr{H}}. \tag{4.12}$$

Proof. of Theorem 4.1. Defining $\mathscr{H}_1$ and $\mathscr{H}_2$ by

$$\mathscr{H}_1 := P_U(L^2_{\mu_k}(\mathbb{R}^{2d})), \quad \mathscr{H}_2 := P_h(L^2_{\mu_k}(\mathbb{R}^{2d})).$$

We proceed as in [16]. We prove that

$$\begin{aligned} ||P_U P_h||_{HS} : &= \Big(\int_{\mathbb{R}^{2d}\times\mathbb{R}^{2d}} |\chi_U(b,x)|^2 |\mathscr{K}_h(b',x';b,x)|^2 d\mu_k(b',x')d\mu_k(b,x)\Big)^{\frac{1}{2}} \\ &\leq \frac{||h||_{L^2_k(\mathbb{R}^d)}}{\sqrt{C_h}}\sqrt{\mu_k(U)} < \infty. \end{aligned} \tag{4.13}$$

Hence, $P_U P_h$ is a Hilbert-Schmidt operator and, therefore it is a compact operator. Now, Lemma 4.1 implies the existence of a constant $C > 0$ such that (4.12) holds for $P_{\mathscr{H}_1} := P_U$ and $P_{\mathscr{H}_2} := P_h$ and the fact that

$$P_{\mathscr{H}_2^{\perp}}(\mathscr{N}_h^D(f)) = (Id - P_h)\mathscr{N}_h^D(f) = 0.$$

This leads to (4.11). □

Definition 4.1. *Let U the subset of $\mathbb{R}^{2d}$ verifying the relation (4.10). Then:*

(1) We say that U is a weakly annihilating set, if any function f in $L^2_k(\mathbb{R}^d)$ vanishes, when its Dunkl wavelet transform $\mathscr{N}_h^D(f)$, is supported in U.

(2) We say that U is a strongly annihilating set, if there exists a constant $C_k(U,h) > 0$ such that for every function f in $L^2_k(\mathbb{R}^d)$, we have

$$C_k(U,h)\|\chi_{U^c}\mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq \|f\|_{L^2_k(\mathbb{R}^d)}. \tag{4.14}$$

The constant $C_k(U,h)$ will be called the annihilation constant of U.

Remark 4.2. *(1) It is clear that, every strongly annihilating set is also a weakly annihilating set.*

(2) From Proposition 4.2, we see that any set $U \subset \mathbb{R}^{2d}$ *with* $\mu_k(U) < \frac{C_h}{||h||^2_{L^2_k(\mathbb{R}^d)}}$*, is strongly annihilating set.*

(3) If $||P_U P_h|| < 1$*, then for all* f *in* $L^2_k(\mathbb{R}^d)$ *we have*

$$\frac{1}{\sqrt{1-||P_U P_h||^2}}\|\chi_{U^c}\mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq \|f\|_{L^2_k(\mathbb{R}^d)}\|h\|_{L^2_k(\mathbb{R}^d)}. \tag{4.15}$$

(4) Following the result established in a general context in [11] p.88, if U *is a strongly annihilating set, we have* $||P_U P_h|| < 1$.

In the following, we give Benedicks-type uncertainty principle for the Dunkl wavelet transform under some condition on the Dunkl wavelet.

Theorem 4.2. *We suppose that the Dunkl wavelet* h *satisfies*

$$\int_{\{\xi\in\mathbb{R}^d:\mathscr{F}_D(h)(\xi)\neq 0\}} d\gamma_k(\xi) < \infty. \tag{4.16}$$

Then for any subset $U \subset \mathbb{R}^{2d}$ *such that for almost every* $b \in \mathbb{R}^d$,

$$\int_{\mathbb{R}^d} \chi_U(b,x)d\gamma_k(x) < \infty,$$

we have

$$P_h(L^2_{\mu_k}(\mathbb{R}^{2d})) \cap P_U(L^2_{\mu_k}(\mathbb{R}^{2d})) = \{0\}. \tag{4.17}$$

Proof. Let F be a non-trivial function in $P_h(L^2_{\mu_k}(\mathbb{R}^{2d}))\cap P_U(L^2_{\mu_k}(\mathbb{R}^{2d}))$, then from the definition of the operators P_U and P_h, there exists a function f in $L^2_k(\mathbb{R}^d)$ such that $F = \mathscr{N}_h^D(f)$ and $supp\, F \subset U$. Let $b \in \mathbb{R}^d$, such that $\int_{\mathbb{R}^d} \chi_U(b,x)d\gamma_k(x) < \infty$. Consider the function $\mathscr{N}_h^D(f)(b,x)$ with respect to the variable x, we denote it $F_b(x)$. Thus, we have

$$supp\, F_b \subset \Big\{x \in \mathbb{R}^d : (b,x) \in U\Big\}$$

and

$$\int_{supp\, F_b} d\gamma_k(x) < \infty.$$

On the other hand using (3.17), (3.8) and the hypothesis (4.16), we deduce that

$$\int_{\{\xi\in\mathbb{R}^d:\mathscr{F}_D(F_b)(\xi)\neq 0\}} d\gamma_k(\xi) < \infty.$$

Using Proposition 2.3, we deduce that for every $x \in \mathbb{R}^d$, $F_b(x) = 0$, which implies that $F = 0$. □

Consequently, we obtain the following improvement.

Corollary 4.1. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ satisfying the relation (4.16). Then for any subset $U \subset \mathbb{R}^{2d}$ verifying the relation (4.10), there exists a constant $C := C_k(h, U) > 0$ such that for all f in $L^2_k(\mathbb{R}^d)$, we have*

$$\|\chi_{U^c} \mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \geq C\|f\|_{L^2_k(\mathbb{R}^d)}. \tag{4.18}$$

We consider a signal G in $L^2_{\mu_k}(\mathbb{R}^{2d})$ which is transmitted to a receiver who knows that G belongs to $\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))$. Suppose that the observation of G is corrupted by a noise n in $L^2_{\mu_k}(\mathbb{R}^{2d})$ (which is nonetheless assumed to be small) and unregistered values on $U \subset \mathbb{R}^{2d}$. Thus, the observable function r satisfies

$$r(a,x) = \begin{cases} G(a,x) + n(a,x) & \text{if} \quad (a,x) \in U^c \\ 0 & \text{if} \quad (a,x) \in U. \end{cases} \tag{4.19}$$

Here we have assumed without loss of generality that $n = 0$ on U. Equivalently, we have

$$r = (Id - P_U)G + n. \tag{4.20}$$

We say that G can be stably reconstructed from r, if there exists a linear operator

$$L_{U,h} : L^2_{\mu_k}(\mathbb{R}^{2d}) \to L^2_{\mu_k}(\mathbb{R}^{2d})$$

and a constant $C_k(h, U)$ such that

$$||G - L_{U,h}(r)||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq C_k(h, U)||n||_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.21}$$

We proceed as in [5], it is easy to prove the following:

Proposition 4.4. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ satisfying the relation (4.16), and let U be a subset of $\mathbb{R}^{2d}$ verifying the relation (4.10). Then G can be stably reconstructed from r. The constant $C(U, h)$ in (4.21) is not larger than $(1 - ||P_U P_h||)^{-1}$.*

Finally using the similar method given in [5], we give an algorithm for computing $L_{U,h}(r)$. The identity

$$L_{U,h} = \sum_{j=0}^{\infty} (P_U P_h)^j$$

suggests an algorithm for computing $L_{U,h}(r)$. Put

$$G_n = \sum_{j=0}^{n} (P_U P_h)^j r,$$

then

$$\begin{cases} G_0 & = & r \\ G_{n+1} & = & r + P_U P_h G_n \end{cases} \tag{4.22}$$

and

$$G_n \to L_{U,h}(r), \quad \text{as } n \to \infty.$$

The iteration converges at a geometric rate to the fixed point

$$G = r + P_U P_h G.$$

Thus, we deduce that

$$G_{n+1} - G \quad = \quad P_U P_h (G_n - G). \tag{4.23}$$

So that, if U is strongly annihilating, then from (4.23), the following error estimate holds:

$$||G_n - G||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq ||P_U P_h||^n ||G - r||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \tag{4.24}$$

and in particular if $\mu_k(U) < \frac{C_h}{||h||^2_{L^2_k(\mathbb{R}^d)}}$, then from (4.13)

$$||G_n - G||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \Big(\frac{||h||_{L^2_k(\mathbb{R}^d)}}{\sqrt{C_h}}\sqrt{\mu_k(U)}\Big)^n ||G - r||_{L^2_{\mu_k}(\mathbb{R}^{2d})}. \tag{4.25}$$

Algorithms of type (4.22), have been applied to a host of problems in signal recovery (see [5]), and others.

5. SPECTRAL ANALYSIS FOR THE GENERALIZED CONCENTRATION OPERATOR

5.1. Preliminaries.

Notation. We denote by

- $(\mathscr{N}_h^D)^* : L^2_{\mu_k}(\mathbb{R}^{2d}) \to L^2_k(\mathbb{R}^d)$ the adjoint of $\mathscr{N}_h^D$ defined by

$$\langle \mathscr{N}_h^D(f), g\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} = C_h \langle f, (\mathscr{N}_h^D)^*(g)\rangle_{L^2_k(\mathbb{R}^d)}, \quad f \in L^2_k(\mathbb{R}^d),\ g \in L^2_{\mu_k}(\mathbb{R}^{2d}). \tag{5.1}$$

- $l^p(\mathbb{N})$ the set of all infinite sequences of real (or complex) numbers $x := (x_j)_{j\in\mathbb{N}}$, such that

$$||x||_p := \Big(\sum_{j=1}^{\infty} |x_j|^p\Big)^{\frac{1}{p}} < \infty, \quad \text{if} \quad 1 \leq p < \infty,$$
$$||x||_\infty := \sup_{j\in\mathbb{N}} |x_j| < \infty.$$

For $p = 2$, we provide this space $l^2(\mathbb{N})$ with the scalar product

$$\langle x, y\rangle_2 := \sum_{j=1}^{\infty} x_j \overline{y_j}.$$

- $B(L^2_k(\mathbb{R}^d))$ the space of bounded operators from $L^2_k(\mathbb{R}^d)$ into itself.

Definition 5.1. *(i) The singular values $(s_n(A))_{n\in\mathbb{N}}$ of a compact operator A in $B(L^2_k(\mathbb{R}^d))$ are the eigenvalues of the positive self-adjoint operator $|A| = \sqrt{A^*A}$.*

(ii) For $1 \leq p < \infty$, the Schatten class S_p is the space of all compact operators whose singular values lie in $l^p(\mathbb{N})$. The space S_p is equipped with the norm

$$||A||_{S_p} := \Big(\sum_{n=1}^{\infty} (s_n(A))^p\Big)^{\frac{1}{p}}. \tag{5.2}$$

Remark 5.1. *We note that S_2 is the space of Hilbert-Schmidt operators, whereas S_1 is the space of trace class operators.*

Definition 5.2. *The trace of an operator A in S_1 is defined by*

$$tr(A) = \sum_{n=1}^{\infty} \langle A v_n, v_n\rangle_{L^2_k(\mathbb{R}^d)} \tag{5.3}$$

where $(v_n)_n$ is any orthonormal basis of $L^2_k(\mathbb{R}^d)$.

Remark 5.2. *If A is positive, then*

$$tr(A) = ||A||_{S_1}. \tag{5.4}$$

Moreover, a compact operator A on the generalized Sobolev space $L^2_k(\mathbb{R}^d)$ is Hilbert-Schmidt, if the positive operator A^*A is in the space of trace class S_1. Then

$$||A||^2_{HS} := ||A||^2_{S_2} = ||A^*A||_{S_1} = tr(A^*A) = \sum_{n=1}^{\infty} ||Av_n||^2_{L^2_k(\mathbb{R}^d)} \tag{5.5}$$

for any orthonormal basis $(v_n)_n$ of $L^2_k(\mathbb{R}^d)$.

Definition 5.3. *We define $S_\infty := B(L^2_k(\mathbb{R}^d))$, equipped with the norm,*

$$||A||_{S_\infty} := \sup_{v\in L^2_k(\mathbb{R}^d):||v||_{L^2_k(\mathbb{R}^d)}=1} ||Av||_{L^2_k(\mathbb{R}^d)}. \tag{5.6}$$

Definition 5.4. *Let h be measurable function on $\mathbb{R}^d$, σ be measurable function on $\mathbb{R}^{2d}$, we define the localization operator noted by $\mathscr{L}_h(\sigma)$, on $L^p_k(\mathbb{R}^d)$, $1 \le p \le \infty$, by*

$$\mathscr{L}_h(\sigma)(f)(y) = \frac{1}{C_h}\int_{\mathbb{R}^{2d}} \sigma(b,x)\mathscr{N}_h^D(f)(b,x)\ h_{b,x}(y)d\mu_k(b,x),\ y \in \mathbb{R}^d. \tag{5.7}$$

It is often more convenient to interpret the definition of $\mathscr{L}_h(\sigma)$ in a weak sense, that is, for f in $L^p_k(\mathbb{R}^d)$, $p \in [1,\infty]$, and g in $L^{p'}_k(\mathbb{R}^d)$,

$$\langle \mathscr{L}_h(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)} = \frac{1}{C_h}\int_{\mathbb{R}^{2d}} \sigma(b,x)\mathscr{N}_h^D(f)(b,x)\overline{\mathscr{N}_h^D(g)(b,x)}d\mu_k(b,x). \tag{5.8}$$

Proposition 5.1. *Let σ in $L^\infty_{\mu_k}(\mathbb{R}^{2d})$, then the localization operator $\mathscr{L}_h(\sigma)$ is in S_∞ and we have*

$$||\mathscr{L}_h(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}. \tag{5.9}$$

Proof. For all functions f and g in $L^2_k(\mathbb{R}^d)$, we have from Hölder's inequality

$$\begin{aligned}|\langle \mathscr{L}_h(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)}| &\leqslant \frac{1}{C_h}\int_0^\infty\int_{\mathbb{R}^d} |\sigma(a,x)||\mathscr{N}_h^D(f)(a,x)||\overline{\mathscr{N}_h^D(g)(a,x)}|d\mu_k(a,x)\\ &\leqslant \frac{1}{C_h}\|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}\|\mathscr{N}_h^D(f)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\|\mathscr{N}_h^D(g)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}.\end{aligned}$$

Using Plancherel's formula for $\mathscr{N}_h^D$, given by the relation (3.19), we get

$$|\langle \mathscr{L}_h(\sigma)(f), g\rangle_{L^2_k(\mathbb{R}^d)}| \leqslant \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}\|f\|_{L^2_k(\mathbb{R}^d)}\|g\|_{L^2_k(\mathbb{R}^d)}.$$

Thus,

$$||\mathscr{L}_h(\sigma)||_{S_\infty} \leqslant \|\sigma\|_{L^\infty_{\mu_k}(\mathbb{R}^{2d})}.$$

□

In this section, h will be a Dunkl wavelet on $\mathbb{R}^d$ such that

$$\|h\|_{L^2_k(\mathbb{R}^d)} = 1.$$

5.2. The range of the Dunkl wavelet transform.
In this section we shall keep our focus on localization operators $\mathscr{L}_h(\sigma)$ with symbol $\sigma = \chi_U$, where U is a subset of $\mathbb{R}^{2d}$ with finite measure $\mu_k(U)$ given by

$$\mu_k(U) := \int_U d\mu_k(a, x). \tag{5.10}$$

In what follows, such operator will be noted by $\mathscr{L}_h(U)$ for the sake of simplicity.

Remark 5.3. *i) We note that $P_h = \mathscr{N}_h^D(\mathscr{N}_h^D)^*$. Explicitly, P_h is the integral operator*

$$P_h F(z) = \int_{\mathbb{R}^{2d}} F(b, x)\mathscr{K}_h(z; b, x)d\mu_k(b, x),\ z = (b', x') \in \mathbb{R}^{2d},$$

with integral kernel $\mathscr{K}_h$ given by (4.1).

ii) As $\mathscr{K}_h$ is the integral kernel of an orthogonal projection, it satisfies

$$\mathscr{K}_h(z; z') = \overline{\mathscr{K}_h(z'; z)}, \quad \text{for all } z, z' \in \mathbb{R}^{2d},$$

and

$$\mathscr{K}_h(z; z') = \int_{\mathbb{R}^{2d}} \mathscr{K}_h(z; z'')\mathscr{K}_h(z''; z')d\mu_k(z''),\ z, z' \in \mathbb{R}^{2d}. \tag{5.11}$$

iii) If $\{v_n : n \in \mathbb{N}\}$ is an orthonormal basis of $\mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$, $\mathscr{K}_h$ can be expanded as

$$\mathscr{K}_h(z; z') = \sum_{n=1}^{\infty} v_n(z)\overline{v_n(z')},\ z, z' \in \mathbb{R}^{2d}. \tag{5.12}$$

Definition 5.5. *Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L_k^2(\mathbb{R}^d)$. We define the Dunkl wavelet scalogram of f as*

$$\mathbf{S}_h^D(f)(b, x) = \frac{1}{C_h}|\mathscr{N}_h^D(f)(b, x)|^2,\ (b, x) \in \mathbb{R}^{2d}. \tag{5.13}$$

Remark 5.4. *From the Plancherel formula of $\mathscr{N}_h^D$, we have*

$$\int_{\mathbb{R}^{2d}} \mathbf{S}_h^D(f)(b, x)d\mu_k(b, x) = \|f\|^2_{L_k^2(\mathbb{R}^d)}. \tag{5.14}$$

It justifies the interpretation of a scalogram as a time-frequency energy density. Note that also by (5.8) we have

$$\langle \mathscr{L}_h(\sigma)f, f\rangle_{L_k^2(\mathbb{R}^d)} = \int_{\mathbb{R}^{2d}} \sigma(b, x)\mathbf{S}_h^D(f)(b, x)d\mu_k(b, x). \tag{5.15}$$

Definition 5.6. *We define the Calderón-Toeplitz operator*

$$T_{h,U} : \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)) \to \mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$$

by

$$T_{h,U} F = P_h P_U F. \tag{5.16}$$

Proposition 5.2. *The operator* $T_{h,U} : \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)) \to \mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$ *is trace-class and satisfies*

$$0 \leq T_{h,U} \leq P_U \leq I, \tag{5.17}$$

and

$$T_{h,U} = \mathscr{N}_h^D \mathscr{L}_h(U)(\mathscr{N}_h^D)^*. \tag{5.18}$$

Proof. For all $F \in \mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$,

$$\begin{aligned} \langle T_{h,U} F, F \rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= \langle P_h(P_U F), F \rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\ &= \langle P_U F, F \rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\ &= \int_U |F(b,x)|^2 d\mu_k(b,x). \end{aligned} \tag{5.19}$$

Thus we deduce (5.17), and $T_{h,U}$ is bounded and positive.

Now, we want to prove (5.18). Indeed, using $\mathscr{N}_h^D$ and $(\mathscr{N}_h^D)^*$, the time-frequency localization operator

$$\mathscr{L}_h(U) : L_k^2(\mathbb{R}^d) \to L_k^2(\mathbb{R}^d)$$

can be written as

$$\mathscr{L}_h(U)(f) = (\mathscr{N}_h^D)^*(P_U \mathscr{N}_h^D f), \quad f \in L_k^2(\mathbb{R}^d).$$

Therefore

$$(\mathscr{N}_h^D \mathscr{L}_h(U)(\mathscr{N}_h^D)^*)F = P_h P_U F = T_{h,U} F, \quad F \in \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)). \tag{5.20}$$

Therefore the time-frequency operator $\mathscr{L}_h(U)$ and the Calderón-Toeplitz operator $T_{h,U}$ are related by

$$T_{h,U} = \mathscr{N}_h^D \mathscr{L}_h(U)(\mathscr{N}_h^D)^*.$$

□

Remark 5.5. *From the above Proposition we deduce that* $T_{h,U}$ *and* $\mathscr{L}_h(U)$ *enjoy the same spectral properties, in particular, we have the following proposition.*

Proposition 5.3. *The Calderón-Toeplitz operator* $T_{h,U}$ *is compact and even trace class with*

$$tr(T_{h,U}) = tr(\mathscr{L}_h(U)) = M_k(h,U), \tag{5.21}$$

where

$$M_k(h,U) := \frac{1}{C_h} \int_U \|h_{b,x}\|^2_{L_k^2(\mathbb{R}^d)} d\mu_k(b,x). \tag{5.22}$$

Proof. We known that the operator $T_{h,U} : \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)) \to \mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$ is bounded and positive. Now, let $\{v_n\}_{n=1}^\infty$ be an arbitrary orthonormal basis for $\mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$. Then if we denote u_n by $u_n = \sqrt{C_h}(\mathscr{N}_h^D)^*(v_n)$, then $\{u_n\}_{n=1}^\infty$ is an orthonormal basis for $L_k^2(\mathbb{R}^d)$.

Thus by (5.8) and Fubini's theorem

$$\begin{aligned}\sum_{n=1}^{\infty}\langle T_{h,U}(v_n),v_n\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= C_h\sum_{n=1}^{\infty}\left\langle \mathscr{L}_h(U)(\mathscr{N}_h^D)^*(v_n),(\mathscr{N}_h^D)^*(v_n)\right\rangle_{L^2_k(\mathbb{R}^d)}\\ &= \tfrac{1}{C_h}\sum_{n=1}^{\infty}\int_U|\mathscr{N}_h^D(u_n)(b,x)|^2d\mu_k(b,x)\\ &= \tfrac{1}{C_h}\int_U\sum_{n=1}^{\infty}|\mathscr{N}_h^D(u_n)(b,x)|^2d\mu_k(b,x)\\ &= \tfrac{1}{C_h}\int_U\sum_{n=1}^{\infty}|\langle u_n,h_{b,x}\rangle_{L^2_k(\mathbb{R}^d)}|^2d\mu_k(b,x)\\ &= \tfrac{1}{C_h}\int_U\|h_{b,x}\|^2_{L^2_k(\mathbb{R}^d)}d\mu_k(b,x)\\ &= M_k(h,U).\end{aligned}$$

Therefore, by Definition 5.2 and Remark 5.2, the operator $T_{h,U}$ is trace class with

$$\|T_{h,U}\|_{S_1}=tr(T_{h,U})=M_k(h,U).$$

□

Let $\mathbf{V}_{h,U}:L^2_{\mu_k}(\mathbb{R}^{2d})\to L^2_{\mu_k}(\mathbb{R}^{2d})$ the operator defined by $\mathbf{V}_{h,U}=P_hP_UP_h$. The advantage of $\mathbf{V}_{h,U}$ compared to $T_{h,U}$ is that it is defined on $L^2_{\mu_k}(\mathbb{R}^{2d})$ and consequently its spectral properties can be easily related to its integral kernel.

Since $T_{h,U}$ is positive and trace-class, then using the decomposition

$$L^2_{\mu_k}(\mathbb{R}^{2d})=\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))\oplus\Big(\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))\Big)^{\perp},$$

we deduce that $\mathbf{V}_{h,U}$ is also positive and trace-class with

$$tr(\mathbf{V}_{h,U})=tr(T_{h,U})=M_k(h,U). \tag{5.23}$$

In addition, we have the following result:

Proposition 5.4. *The trace of $T^2_{h,U}$ is given by*

$$tr(T^2_{h,U})=\int_U\int_U|\mathscr{K}_h(a,x;b',x')|^2d\mu_k(b,x)d\mu_k(b',x'). \tag{5.24}$$

Proof. As $\mathbf{V}_{h,U}$ is positive, then

$$tr(T^2_{h,U})=tr(\mathbf{V}^2_{h,U}). \tag{5.25}$$

On the other hand using the fact that the space $\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))$ is a reproducing kernel Hilbert space with kernel $\mathscr{K}_h$, we get that for $F\in L^2_{\mu_k}(\mathbb{R}^{2d})$

$$\begin{aligned}&\mathbf{V}_{h,U}F(b,x)=\\ &\int_{\mathbb{R}^{2d}}F(b',x')\int_{\mathbb{R}^{2d}}\chi_U(b,y)\mathscr{K}_h(b,x;b,y)\mathscr{K}_h(b,y;b',x')d\mu_k(b,y)d\mu_k(b',x').\end{aligned} \tag{5.26}$$

That is, $\mathbf{V}_{h,U}$ has integral kernel

$$\mathbf{N}_{h,U}(b,x;b',x')=\int_{\mathbb{R}^{2d}}\chi_U(b,y)\mathscr{K}_h(b,x;b,y)\mathscr{K}_h(b,y;b',x')d\mu_k(b,y). \tag{5.27}$$

Therefore

$$
\begin{aligned}
tr(\mathbf{V}_{h,U}^2) &= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} |\mathbf{N}_{h,U}(b,x;b',x')|^2 d\mu_k(b,x)d\mu_k(b',x') \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \mathbf{N}_{h,U}(b,x;b',x')\overline{\mathbf{N}_{h,U}(b,x;b',x')}d\mu_k(b,x)d\mu_k(b',x') \\
&= \int_{\mathbb{R}^{2d}}\int_{\mathbb{R}^{2d}} \chi_{_U}(z_1)\chi_{_U}(z_2)\mathbf{K}_h(z_1;z_2)d\mu_k(z_1)d\mu_k(z_2)
\end{aligned}
$$

where by using the properties of the kernel of the reproducing kernel Hilbert space, we derive that

$$\mathbf{K}_h(z_1;z_2) \quad = \quad \mathscr{K}_h(z_2;z_1)\mathscr{K}_h(z_1;z_2).$$

Using (5.11), we get

$$\mathbf{K}_h(z_1;z_2) = |\mathscr{K}_h(z_1;z_2)|^2. \tag{5.28}$$

This follows us to conclude. □

5.3. Eigenvalues and eigenfunctions.

Since the localization operator $\mathscr{L}_h(U) = (\mathscr{N}_h^D)^*\chi_{_U}\mathscr{N}_h^D$ that we consider is a compact and self-adjoint operator, the spectral theorem gives the following spectral representation

$$\mathscr{L}_h(U)(f) = \sum_{n=1}^{\infty} s_n(U)\left\langle f, \varphi_n^U\right\rangle_{L_k^2(\mathbb{R}^d)}\varphi_n^U, \quad f \in L_k^2(\mathbb{R}^d), \tag{5.29}$$

where $\{s_n(U)\}_{n=1}^{\infty}$ are the positive eigenvalues arranged in a nonincreasing manner and $\{\varphi_n^U\}_{n=1}^{\infty}$ is the corresponding orthonormal set of eigenfunctions. Note that $s_n(U) \searrow 0$ and by (5.9), we have for all $n \geq 1$,

$$s_n(U) \leq s_1(U) \leq 1. \tag{5.30}$$

This, together with (5.18), we can deduce that the Toeplitz operator

$$T_{h,U} : \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)) \to \mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$$

can be diagonalized as

$$T_{h,U}\, F = \sum_{n=1}^{\infty} s_n(U)\left\langle F, t_n^U\right\rangle_{L_{\mu_k}^2(\mathbb{R}^{2d})} t_n^U, \quad F \in \mathscr{N}_h^D(L_k^2(\mathbb{R}^d)), \tag{5.31}$$

where $t_n^U = \frac{1}{\sqrt{C_h}}\mathscr{N}_h^D(\varphi_n^U)$.

Lemma 5.1. *For all $z = (b,x) \in \mathbb{R}^{2d}$, we have*

$$\Theta(z) := \int_{\mathbb{R}^{2d}} \chi_{_U}(\omega)|\mathscr{K}_h(\omega;z)|^2 d\mu_k(\omega) = \sum_{n=1}^{\infty} s_n(U)\mathbf{S}_h^D(\varphi_n^U)(z). \tag{5.32}$$

Proof. From Proposition 4.1, we have for all $z = (b,x) \in \mathbb{R}^{2d}$, the function $\mathscr{K}_h(.;z)$ is in $\mathscr{N}_h^D(L_k^2(\mathbb{R}^d))$. Therefore using the properties of the kernel of the reproducing kernel Hilbert space, we get

$$\begin{aligned}\langle T_{h,U}\,\mathscr{K}_h(.;z),\mathscr{K}_h(.;z)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= \langle P_U\mathscr{K}_h(.;z),\mathscr{K}_h(.;z)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}\\ &= \int_{\mathbb{R}^{2d}}\chi_{_U}(\omega)\mathscr{K}_h(\omega;z)\overline{\mathscr{K}_h(\omega;z)}d\mu_k(\omega)\\ &= \int_{\mathbb{R}^{2d}}\chi_{_U}(\omega)|\mathscr{K}_h(\omega;z)|^2d\mu_k(\omega).\end{aligned}$$

Let $\{w_n^U\}_{n=1}^{\infty}\subset\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))$ be an orthonormal basis of $Ker(T_{h,U})$ (eventually empty). Hence, $\{t_n^U\}_{n=1}^{\infty}\cup\{w_n^U\}_{n=1}^{\infty}$ is an orthonormal basis of $\mathscr{N}_h^D(L^2_k(\mathbb{R}^d))$ and therefore the reproducing kernel $\mathscr{K}_h$ can be written as

$$\mathscr{K}_h(b,x;b',x')=\overline{\mathscr{K}_h(b',x';z)}=\sum_{n=1}^{\infty}t_n^U(z)\overline{t_n^U(b',x')}+\sum_{n=1}^{\infty}w_n^U(z)\overline{w_n^U(b',x')}.\quad(5.33)$$

Using this, we compute again

$$\begin{aligned}\langle T_{h,U}\,\mathscr{K}_h(.;z),\mathscr{K}_h(.;z)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= \left\langle T_{h,U}\sum_{n=1}^{\infty}\overline{t_n^U(z)}t_n^U,\sum_{m=1}^{\infty}\overline{t_m^U(z)}t_m^U\right\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}\\ &= \sum_{n,m}\overline{t_n^U(z)}t_m^U(z)\left\langle T_{h,U}\,t_n^U,t_m^U\right\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}\\ &= \sum_{n=1}^{\infty}s_n(U)|t_n^U(z)|^2,\end{aligned}$$

and the conclusion follows. □

Let $\varepsilon\in(0,1)$ and define the quantity

$$n(\varepsilon,U):=card\Big\{j:s_j(U)\geq 1-\varepsilon\Big\}.\quad(5.34)$$

Then an easy adaptation of the proof of Lemma 3.3 in [1], we obtain the following estimate for the eigenvalue distribution.

Proposition 5.5. *Let $\varepsilon\in(0,1)$. We have*

$$\begin{aligned}&|n(\varepsilon,U)-M_k(h,U)|\leq max\{\tfrac{1}{\varepsilon},\tfrac{1}{1-\varepsilon}\}\\ &\Big|\tfrac{1}{C_h}\int_U\int_U|\mathscr{K}_h(b',x';b,x)|^2d\mu_k(b,x)d\mu_k(b',x')-M_k(h,U)\Big|.\end{aligned}$$

5.4. Scalogram of a subspace. Given an N-dimentional subspace V of $L^2_k(\mathbb{R}^d)$, P_V the orthogonal projection onto V with projection kernel $\mathscr{G}_V$, i.e.

$$P_Vf(.)=\int_{\mathbb{R}^d}\mathscr{G}_V(.,t)f(t)d\gamma_k(t).\quad(5.35)$$

Recall that if $\{v_n\}_{n=1}^N$ is an orthonormal basis of V, then

$$\mathscr{G}_V(x,t)=\sum_{n=1}^{N}v_n(x)\overline{v_n(t)}.\quad(5.36)$$

The kernel $\mathscr{G}_V$ is independent of the choice of orthonormal basis for V.

Definition 5.7. *The scalogram of the space V with Dunkl wavelet h is defined*

$$\mathbf{SCAL}_h^D V(b,x) := \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \mathscr{G}_V(t,y)\overline{h_{b,x}(t)}h_{b,x}(y)d\gamma_k(t)d\gamma_k(y). \tag{5.37}$$

Then we have the following result:

Lemma 5.2. *The scalogram $\mathbf{SCAL}_h^D V$ is given by*

$$\mathbf{SCAL}_h^D V = C_h \sum_{n=1}^{N} \mathbf{S}_h^D(v_n). \tag{5.38}$$

Proof. We have

$$\begin{aligned}
\mathbf{SCAL}_h^D V(b,x) &= \int_{\mathbb{R}^d}\int_{\mathbb{R}^d}\sum_{n=1}^{N} v_n(t)\overline{v_n(y)h_{b,x}(t)}h_{b,x}(y)d\gamma_k(t)d\gamma_k(y)\\
&= \sum_{n=1}^{N}\langle v_n, h_{b,x}\rangle_{L_k^2(\mathbb{R}^d)}\overline{\langle v_n, h_{b,x}\rangle}_{L_k^2(\mathbb{R}^d)}\\
&= \sum_{n=1}^{N}\mathscr{N}_h^D(v_n)(b,x)\overline{\mathscr{N}_h^D(v_n)(b,x)}\\
&= \sum_{n=1}^{N}|\mathscr{N}_h^D(v_n)(b,x)|^2.
\end{aligned}$$

This allows us to conclude. □

Definition 5.8. *We define the time-frequency concentration of a subspace V in U as:*

$$\xi_{U,h}(V) := \frac{1}{N}\int_U \mathbf{SCAL}_h^D V(b,x)d\mu_k(b,x). \tag{5.39}$$

Then from Lemma 5.2,

$$\xi_{U,h}(V) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^D(v_n)(b,x)d\mu_k(b,x). \tag{5.40}$$

Theorem 5.1. *The N-dimentional signal space $V_N = span\{\varphi_n^U\}_{n=1}^N$ consisting of the first N eigenfunctions of $\mathscr{L}_h(U)$ corresponding to the N largest eigenvalues $\{s_n(U)\}_{n=1}^N$ maximize the regional concentration $\xi_{U,h}(V)$ and*

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{5.41}$$

Proof. We have

$$\xi_{U,h}(V_N) := \frac{C_h}{N}\sum_{n=1}^{N}\int_U \mathbf{S}_h^D(\varphi_n^U)(b,x)d\mu_k(b,x). \tag{5.42}$$

Moreover, the min-max lemma for self-adjoint operators states that (see e. g. Sec.95 in [23])

$$\begin{aligned} s_n(U) &= \int_U \mathbf{S}_h^D(\varphi_n^U)(z)d\mu_k(b,x) \\ &= \max\Big\{ \langle \mathscr{L}_h(U)(f), f\rangle_{L_k^2(\mathbb{R}^d)} : \|f\|_{L_k^2(\mathbb{R}^d)} = 1, f \perp \varphi_1^U, ..., \varphi_{n-1}^U \Big\}. \end{aligned}$$

So the eigenvalues of $\mathscr{L}_h(U)$ determine the number of orthogonal functions that have a well-concentrated scalogram in U. Thus,

$$\xi_{U,h}(V_N) = \frac{C_h}{N}\sum_{n=1}^{N} s_n(U). \tag{5.43}$$

The min-max characterization of the eigenvalues of compact operators implies that the first N eigenfunctions of the time-frequency operator $\mathscr{L}_h(U)$ have optimal cumulative time-frequency concentration inside U, in the sense,

$$\sum_{n=1}^{N} \langle \mathscr{L}_h(U)(\varphi_n^U), \varphi_n^U\rangle_{L_k^2(\mathbb{R}^d)} = \max\Big\{ \sum_{n=1}^{N} \langle \mathscr{L}_h(U)v_n, v_n\rangle_{L_k^2(\mathbb{R}^d)} : \|v_n\|_{L_k^2(\mathbb{R}^d)} = 1\Big\}. \tag{5.44}$$

Therefore any N-dimensional subset V of $L_k^2(\mathbb{R}^d)$ cannot to be better concentrated in U than V_N, i.e

$$\xi_{U,h}(V) \leq \xi_{U,h}(V_N). \tag{5.45}$$

The proof is complete. □

Remark 5.6. *The time-frequency concentration of a subspace V_N in U satisfies,*

$$s_N(U) \leq C_h^{-1}\xi_{U,h}(V_N) \leq s_1(U) \leq 1. \tag{5.46}$$

5.5. Accumulated scalogram. Let $\rho_{(h,U)} := \mathbf{SCAL}_h^k V_{N_k(h,U)}$, called the accumulated scalogram, where we assume that $N_k(h,U) = [M_k(h,U)]$ is the smallest integer greater than or equal to $M_k(h,U)$ and

$$V_{N_k(h,U)} = span\{v_n^U\}_{n=1}^{N_k(h,U)}.$$

Then

$$\rho_{(h,U)}(b,x) = \sum_{n=1}^{N_k(h,U)} |\mathscr{N}_h^D(v_n^U)(b,x)|^2 = \sum_{n=1}^{N_k(h,U)} |\varphi_n^U(b,x)|^2. \tag{5.47}$$

Note that

$$\|\rho_{(h,U)}\|_{L^1_{\mu_k}(\mathbb{R}^{2d})} = C_h N_k(h,U) = C_h M_k(h,U) + O(1).$$

Moreover, since

$$\sum_{n=1}^{N_k(h,U)} s_n(U) \leq tr(\mathscr{L}_h(U)) = M_k(h,U)$$

then we can define the quantity

$$E(h,U) := 1 - \frac{\sum_{n=1}^{N_k(h,U)} s_n(U)}{M_k(h,U)} \tag{5.48}$$

which satisfies,

$$0 \leq E(h,U) \leq 1. \tag{5.49}$$

More precisely, we have the following result:

Lemma 5.3. *Let $\varepsilon \in (0,1)$. We have*

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\min(1, \frac{n(\varepsilon,U)}{M_k(h,U)}). \tag{5.50}$$

Proof. Let $\varepsilon \in (0,1)$ and define $l_k(\varepsilon,U) = \min(N_k(h,U), n(\varepsilon,U))$. It follows that

$$s_n(U) \geq 1-\varepsilon, \quad 1 \leq n \leq l_k(\varepsilon,U). \tag{5.51}$$

As $N_k(h,U) \geq l_k(h,U)$, we get

$$\sum_{n=1}^{N_k(h,U)} s_n(U) \geq \sum_{n=1}^{l_k(\varepsilon,U)} s_n(U) \geq (1-\varepsilon) l_k(\varepsilon,U). \tag{5.52}$$

Therefore

$$0 \leq E(h,U) \leq 1 - (1-\varepsilon)\frac{l_k(\varepsilon,U)}{M_k(h,U)}. \tag{5.53}$$

As $N_k(\varepsilon,U) \geq M_k(h,U)$, we obtain the desired result. □

Consequently when the eigenvalues $\{s_n(U)\}_{n=0}^{n(\varepsilon,U)}$ are close to 1, then we have $E(h,U) \to 0$. Moreover, we have the following result bounding the error between $\rho_{(h,U)}$ and Θ.

Proposition 5.6. *We have*

$$\frac{1}{M_k(h,U)}\|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{C_h}{M_k(h,U)} + 2C_h E(h,U). \tag{5.54}$$

Proof. From Lemma 5.1, we have, for all $z = (b,x) \in U$

$$\rho_{(h,U)}(z) - C_h\Theta(z) = \sum_{n=1}^{\infty}(t_n - s_n(U))|\varphi_n^U(z)|^2, \tag{5.55}$$

where $t_n = 1$ if $n \leq N_k(h,U)$ and 0 otherwise. As

$$\||\varphi_n^U|^2\|_{L^1_{\mu_k}(\mathbb{R}^{2d})} = C_h \quad \text{and} \quad \sum_{n=1}^{\infty} s_n(U) = M_k(h,U),$$

we get

$$\begin{aligned}
\|\rho_{(h,U)} - C_h\Theta\|_{L^1_{\mu_k}(\mathbb{R}^{2d})} &\leq C_h \sum_{n=1}^{\infty} |t_n - s_n(U)| \\
&= C_h \sum_{n=1}^{N_k(h,U)} (1 - s_n(U)) + C_h \sum_{n>N_k(h,U)} s_n(U) \\
&= C_h N_k(h,U) + C_h \sum_{n=1}^{\infty} s_n(U) - 2C_h \sum_{n=1}^{N_k(h,U)} s_n(U) \\
&= C_h N_k(h,U) + C_h M_k(h,U) - 2C_h \sum_{n=1}^{N_k(h,U)} s_n(U) \\
&\leq C_h + 2C_h\Big(M_k(h,U) - \sum_{n=1}^{N_k(h,U)} s_n(U)\Big),
\end{aligned}$$

and the estimate (5.54) follows. □

6. MEAN DISPERSION THEOREM FOR THE DEFORMED WAVELET TRANSFORM

In this section we will assume that h is a fixed function in $L^2_k(\mathbb{R}^d)$ such that $C_h = 1$.

Definition 6.1. *Let $0 < \varepsilon < 1$ and $U \subset \mathbb{R}^{2d}$ be a measurable subset. Let h be a Dunkl wavelet and $f \in L^2_k(\mathbb{R}^d)$. We say that $\mathscr{N}_h^D(f)$ is ε concentrated on U, if*

$$\Big|\Big| \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(U^c)} \leq \varepsilon \Big|\Big| \mathscr{N}_h^D(f)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}.$$

Proposition 6.1. *Let h be a Dunkl wavelet and $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$ and U be a measurable subset of $\mathbb{R}^{2d}$. If $\mu_k(U) < \infty$, then for every nonempty finite subset $\mathscr{K} \subset \mathbb{N}^d$, we have*

$$\sum_{\nu\in\mathscr{K}} \Big(1 - \|\chi_{U^c}\mathscr{N}_h^D(\varphi_\nu)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big) \leqslant |h\|^2_{L^2_k(\mathbb{R}^d)}\mu_k(U).$$

Proof. Let $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$, by relation (3.19) we deduce that $\frac{1}{\sqrt{C_h}}\left(\mathscr{N}_h^D(\varphi_\nu)\right)_{\nu\in\mathbb{N}^d}$ is an orthonormal sequence in $L^2_{\mu_k}(\mathbb{R}^{2d})$. Moreover, as $P_U P_h$ is an Hilbert-Schmidt operator then by (5.5) and (5.3), it is easy to see that

$$\begin{aligned}
\sum_{\nu\in\mathscr{K}} \langle P_U\mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= \sum_{\nu\in\mathscr{K}} \langle P_h P_U P_h \mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\
&\leq tr(P_h P_U P_h) \\
&= \|P_U P_h\|^2_{HS}.
\end{aligned}$$

Then by (4.13) we get

$$\sum_{\nu\in\mathscr{K}} \langle P_U\mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leqslant |h\|^2_{L^2_k(\mathbb{R}^d)}\mu_k(U). \tag{6.1}$$

Now by Cauchy-Schwarz inequality we have for every $\nu \in \mathscr{K}$,

$$\begin{aligned}\langle P_U \mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} &= 1 - \langle P_{U^c} \mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} \\ &\geqslant 1 - \|\chi_{U^c} \mathscr{N}_h^D(\varphi_\nu)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\end{aligned}$$

in particular by relation (6.1)

$$\sum_{\nu \in \mathscr{K}} \Big(1 - \|\chi_{U^c} \mathscr{N}_h^D(\varphi_\nu)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big) \leqslant \sum_{\nu \in \mathscr{K}} \langle P_U \mathscr{N}_h^D(\varphi_\nu), \mathscr{N}_h^D(\varphi_\nu)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$
$$\leqslant \|h\|^2_{L^2_k(\mathbb{R}^d)} \mu_k(U).$$

□

Then by Proposition 6.1, we shall deduce the following Proposition which shows that, if the Dunkl wavelet continuous transform of an othornormal sequence are ε concentrated on a given centered ball of $\mathbb{R}^{2d}$, then such sequence is necessary finite.

Proposition 6.2. *Let ε and δ be positive real numbers such that $0 < \varepsilon < 1$, and h be a Dunkl wavelet . Let $\mathscr{K} \subset \mathbb{N}^d$ be a nonempty subset and $(\varphi_\nu)_{\nu \in \mathscr{K}}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. If $\mathscr{N}_h^D(\varphi_\nu)$ is ε concentrated on the ball $V_\delta := \{(b,x) \in \mathbb{R}^{2d} : ||(b,x)|| \leq \delta\}$, for every $\nu \in \mathscr{K}$, then $\mathscr{K}$ is finite and*

$$\mathrm{Card}(\mathscr{K}) \leqslant \frac{M(d,h)}{1-\varepsilon} \delta^{4\gamma+2d}, \tag{6.2}$$

where

$$M(d,h) = \mu_k(V_1) \|h\|^2_{L^2_k(\mathbb{R}^d)}.$$

Proof. Let $\mathscr{M} \subset \mathscr{K}$ be a nonempty finite subset, then by Proposition 6.1, we deduce that

$$\sum_{\nu \in \mathscr{M}} \Big(1 - \|\chi_{V_\delta^c} \mathscr{N}_h^D(\varphi_\nu)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})}\Big) \leqslant \|h\|^2_{L^2_k(\mathbb{R}^d)} \mu_k(V_\delta), \tag{6.3}$$

however for every $\nu \in \mathscr{M}$, $\|\chi_{V_\delta^c} \mathscr{N}_h^D(\varphi_\nu)\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leqslant \varepsilon$, and

$$\mu_k(V_\delta) = \mu_k(V_1) \delta^{4\gamma+2d}, \tag{6.4}$$

hence by combining relations (6.3) and (6.4), we deduce that

$$\mathrm{Card}(\mathscr{M}) \leqslant \frac{\|h\|^2_{L^2_k(\mathbb{R}^d)} \mu_k(V_1)}{1-\varepsilon} \delta^{4\gamma+2d},$$

which means that $\mathscr{K}$ is finite and satisfies relation (6.2). □

Let p be a positive real number, h be a Dunkl wavelet and $f \in L^2_k(\mathbb{R}^d)$, we define the Dunkl wavelet p^{th} time-frequency dispersion of $\mathscr{N}_h^D(f)$ by

$$\rho_p(\mathscr{N}_h^D(f)) = \left(\int_{\mathbb{R}^{2d}} ||(b,x)||^p \left|\mathscr{N}_h^D(f)(b,x)\right|^2 d\mu_k(b,x) \right)^{\frac{1}{p}}.$$

Corollary 6.1. *Let A, p be positive real numbers and $h \in L^2_k(\mathbb{R}^d)$ be a Dunkl wavelet. Let $\mathscr{K} \subset \mathbb{N}^d$ be a nonempty subset and $(\varphi_\nu)_{\nu\in\mathscr{K}}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. Assume that for every $\nu \in \mathscr{K}$,*

$$\rho_p(\mathscr{N}_h^D(\varphi_\nu)) \leqslant A,$$

then $\mathscr{K}$ is finite and

$$\mathrm{Card}(\mathscr{K}) \leqslant M'(d,p,h)A^{4\gamma+2d},$$

where $M'(d,p,h) = 2^{1+\frac{8\gamma+4d}{p}}M(d,h)$.

Proof. Assume that $\rho_p(\mathscr{N}_h^D(\varphi_\nu)) \leqslant A$ for every $\nu \in \mathscr{K}$, then we have

$$\int_{V^c_{A2^{\frac{2}{p}}}} |\mathscr{N}_h^D(\varphi_\nu)(b,x)|^2 d\mu_k(b,x) \leqslant \frac{1}{\left(A2^{\frac{2}{p}}\right)^p}\rho_p^p(\mathscr{N}_h^D(\varphi_\nu)) \leqslant \frac{1}{4}. \tag{6.5}$$

Relation (6.5) means that for every $\nu \in \mathscr{K}$, $\mathscr{N}_h^D(\varphi_\nu)$ is $\frac{1}{2}$-concentrated in the set $V_{A2^{\frac{2}{p}}}$, hence according to Proposition 6.2, we deduce that $\mathscr{K}$ is finite and

$$\mathrm{Card}(\mathscr{K}) \leqslant M'(d,p,h)A^{4\gamma+2d}.$$

□

Lemma 6.1. *Let h be a Dunkl wavelet and p be a positive real number. If $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ is an orthonormal sequence in $L^2_k(\mathbb{R}^d)$, then there exists $j_0 \in \mathbb{Z}$ such that*

$$\forall \nu \in \mathbb{N}^d,\ \rho_p(\mathscr{N}_h^D(\varphi_\nu)) \geqslant 2^{j_0-1}.$$

Proof. The proof is immediate consequence of Heisenberg-type inequality (4.6). □

Theorem 6.1 (Shapiro's dispersion theorem)**.** *Let h be a Dunkl wavelet and $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$, then for every positive real number p there is a positive constant C such that for every nonempty finite subset $\mathscr{K} \subset \mathbb{N}^d$, we have*

$$\sum_{\nu\in\mathscr{K}} \left(\rho_p(\mathscr{N}_h^D(\varphi_\nu))\right)^p \geqslant \left(\frac{3}{M'(d,p,h)2^{8\gamma+4d+1}}\right)^{\frac{p}{4\gamma+2d}} (\mathrm{Card}(\mathscr{K}))^{1+\frac{p}{4\gamma+2d}}. \tag{6.6}$$

Proof. For every $j \in \mathbb{Z}$ let

$$P_j = \left\{\nu \in \mathbb{N}^d\ :\ \rho_p(\mathscr{N}_h^D(\varphi_\nu)) \in [2^{j-1}, 2^j)\right\},$$

then for every $\nu \in P_j$

$$\int_{\mathbb{R}^{2d}} ||(b,x)||^p \left|\mathscr{N}_h^D(\varphi_\nu)(b,x)\right|^2 d\mu_k(b,x) \leqslant 2^j,$$

hence, identical from the relation (6.5) we have

$$\int_{V^c_{2^{j+\frac{2}{p}}}} \left|\mathscr{N}_h^D(\varphi_\nu)(b,x)\right|^2 d\mu_k(b,x) \leqslant \tfrac{1}{4}\tfrac{\rho_p(\mathscr{N}_h^D(\varphi_\nu))^p}{2^{jp}} \leqslant \tfrac{1}{4} \tag{6.7}$$

and therefore by mean of relation (6.7), we deduce that for every $\nu \in P_j$, $\mathscr{N}_h^D(\varphi_\nu)$ is $\frac{1}{2}$-concentrated in the ball $V_{2^{j+\frac{2}{p}}}$, on the other words the sequence $(\varphi_\nu)_{\nu\in P_j}$ satisfies the conditions of Corollary 6.1, which shows that P_j is finite and

$$\mathrm{Card}(P_j) \leqslant M'(d,p,h)2^{j(4\gamma+2d)}. \tag{6.8}$$

For $m \in \mathbb{Z},\ m \geqslant j_0$, we denote by $Q_m = \bigcup_{j=j_0}^{m} P_j$ then according to relation (6.8), we have

$$\mathrm{Card}\,(Q_m) = \sum_{j=j_0}^{m} \mathrm{Card}(P_j) \leqslant \frac{M'(d,p,h)2^{4\gamma+2d}}{3} 2^{m(4\gamma+2d)}.$$

Now if $\mathrm{Card}(\mathscr{K}) > \frac{M'(d,p,h)2^{4\gamma+2d+1}}{3} 2^{j_0(4\gamma+2d)}$, then we can choose an integer $n > j_0$ such that

$$\frac{M'(d,p,h)2^{4\gamma+2d+1}}{3} 2^{(n-1)(4\gamma+2d)} < \mathrm{Card}(\mathscr{K}) \leqslant \frac{M'(d,p,h)2^{4\gamma+2d+1}}{3} 2^{n(4\gamma+2d)}. \tag{6.9}$$

Thus by relation (6.9) we get

$$\begin{aligned}\sum_{\nu\in\mathscr{K}} \left(\rho_p(\mathscr{N}_h^D(\varphi_\nu))\right)^p &\geqslant \tfrac{\mathrm{Card}(\mathscr{K})}{2} 2^{(n-1)p} \\ &\geqslant \tfrac{1}{2}\left(\mathrm{Card}(\mathscr{K})\right)^{1+\frac{p}{4\gamma+2d}} \left(\tfrac{3}{2^{8\gamma+4d+1}M'(d,p,h)}\right)^{\frac{p}{4\gamma+2d}}.\end{aligned}$$

Finally, if $\mathrm{Card}(\mathscr{K}) \leqslant \frac{M'(d,p,h)2^{4\gamma+2d+1}}{3} 2^{j_0(4\gamma+2d)}$, then

$$\begin{aligned}\sum_{\nu\in\mathscr{K}} \left(\rho_p(\mathscr{N}_h^D(\varphi_\nu))\right)^p &\geqslant \mathrm{Card}(\mathscr{K})2^{(j_0-1)p} \\ &\geqslant \mathrm{Card}(\mathscr{K})^{1+\frac{p}{4\gamma+2d}} \left(\tfrac{3}{M'(d,p,h)2^{8\gamma+4d+1}}\right)^{\frac{p}{4\gamma+2d}}.\end{aligned}$$

□

Corollary 6.2. *Let $p > 0$, h be a Dunkl wavelet and let $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. Then for every $\mathscr{K} \subset \mathbb{N}^d$*

$$\begin{aligned}&\sum_{\nu\in\mathscr{K}} \left(\left|\left| \, ||b||^{\frac{p}{2}} \mathscr{N}_h^D(\varphi_\nu)(b,x) \right|\right|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} + \left|\left| \, ||x||^{\frac{p}{2}} \mathscr{N}_h^D(\varphi_\nu)(b,x) \right|\right|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})} \right) \\ &\geq \left(\tfrac{3}{M'(d,p,h)2^{12\gamma+6d+1}}\right)^{\frac{p}{4\gamma+2d}} \mathrm{Card}(\mathscr{K})^{1+\frac{p}{4\gamma+2d}}.\end{aligned}$$

Proof. The result is immediately from the previous theorem and the fact that

$$||(b,x)||^p \leq 2^p(||b||^p + ||x||^p).$$

□

The last dispersion inequality implies in particular that, there does not exist an infinite sequence $(\varphi_\nu)_{\nu\in\mathscr{K}}$ in $L^2_k(\mathbb{R}^d)$ such that the two sequences

$$\Big|\Big|\ ||b||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \quad \text{and} \quad \Big|\Big|\ ||x||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|_{L^2_{\mu_k}(\mathbb{R}^{2d})}$$

are bounded. More precisely:

Corollary 6.3. *Let $p>0$, h be a Dunkl wavelet and let $(\varphi_\nu)_{\nu\in\mathbb{N}^d}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$. Then for every $\mathscr{K}\subset\mathbb{N}^d$*

$$\sup_{\nu\in\mathscr{K}}\left(\Big|\Big|\ ||b||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})},\Big|\Big|\ ||x||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}\right)$$
$$\geq\left(\frac{3}{M'(d,p,h)2^{12\gamma+6d+1}}\right)^{\frac{p}{4\gamma+2d}}\mathrm{Card}(\mathscr{K})^{1+\frac{p}{4\gamma+2d}}.$$

In particular

$$\sup_{\nu\in\mathbb{N}^d}\left(\Big|\Big|\ ||b||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}+\Big|\Big|\ ||x||^{\frac{p}{2}}\mathscr{N}_h^D(\varphi_\nu)(b,x)\Big|\Big|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}\right)=\infty.$$

Theorem 6.2 (Shapiro's umbrella theorem)**.** *Let h be a Dunkl wavelet, $\mathscr{K}\subset\mathbb{N}^d$ be a nonempty subset and let $(\varphi_\nu)_{\nu\in\mathscr{K}}$ be an orthonormal sequence in $L^2_k(\mathbb{R}^d)$, if there is a function $g\in L^2_{\mu_k}(\mathbb{R}^{2d})$ such that*

$$|\mathscr{N}_h^D(\varphi_\nu)(b,x)|\leqslant|g(b,x)|,$$

for every $\nu\in\mathscr{K}$ and for almost every $(b,x)\in\mathbb{R}^{2d}$, then $\mathscr{K}$ is finite.

Proof. Following the idea of Malinnikova [13], for every positive real $0<\varepsilon<1$, there is a subset $\Delta_{g,\varepsilon}\subset\mathbb{R}^{2d}$ such that

$$\mu_k(\Delta_{g,\varepsilon})=\inf\left\{\mu_k(U)\ :\ \int\int_{\mathbb{R}^{2d}\setminus U}|g(b,x)|^2\,d\mu_k(b,x)\leqslant\varepsilon^2\right\},$$

and

$$\int\int_{\mathbb{R}^{2d}\setminus\Delta_{g,\varepsilon}}|g(b,x)|^2\,d\mu_k(b,x)=\varepsilon^2.$$

Hence, according to the hypothesis, for every $\nu\in\mathscr{K}$ we have

$$\int\int_{\mathbb{R}^{2d}\setminus\Delta_{g,\varepsilon}}\left|\mathscr{N}_h^D\left(\varphi_\nu\right)(b,x)\right|^2d\mu_k(b,x)\leqslant\varepsilon^2,$$

and by the Proposition 6.1, we get $\mathrm{Card}(\mathscr{K})(1-\varepsilon)\leqslant\mu_k(\Delta_{g,\varepsilon})$. □

7. EXTREMAL FUNCTIONS ON $W^s_k(\mathbb{R}^d)$

The results of this section are in sprit of the following references (cf. [19, 27, 28]).

Definition 7.1. *([14]). Let $s\in\mathbb{R}$, we define the generalized Sobolev space $W^s_k(\mathbb{R}^d)$ as*

$$\left\{u\in\mathscr{S}'(\mathbb{R}^d):(1+||\xi||^2)^{\frac{s}{2}}\mathscr{F}_D(u)\in L^2_k(\mathbb{R}^d)\right\}.$$

We provided this space with inner product $\langle .,. \rangle_{W^s_k(\mathbb{R}^d)}$ given by:

$$\langle f, g\rangle_{W^s_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} (1+||\xi||^2)^s \mathscr{F}_D(f)(\xi)\overline{\mathscr{F}_D(g)(\xi)} d\gamma_k(\xi), \quad \text{for all } f, g \in W^s_k(\mathbb{R}^d). \tag{7.1}$$

Proposition 7.1. *For $s > \frac{d+2\gamma}{2}$, the generalized Sobolev space $W^s_k(\mathbb{R}^d)$ admits the following reproducing kernel:*

$$\mathfrak{K}_s(x, y) = \int_{\mathbb{R}^d} \frac{K(i\xi, x)K(-i\xi, y)d\gamma_k(\xi)}{(1+||\xi||^2)^s}$$

that is

(i) For all $y \in \mathbb{R}^d$, the function $x \mapsto \mathfrak{K}_s(x, y)$ belongs to $W^s_k(\mathbb{R}^d)$.

(ii) The reproducing property: for all $f \in W^s_k(\mathbb{R}^d)$ and $y \in \mathbb{R}^d$,

$$f(y) = \langle f, \mathfrak{K}_s(x, y)\rangle_{W^s_k(\mathbb{R}^d)}.$$

Proof. i) Let y be in $\mathbb{R}^d$. It is easy to see that the function

$$\Upsilon_y : \xi \mapsto \frac{K(-i\xi, y)}{(1+||\xi||^2)^s}$$

belongs to $L^1_k(\mathbb{R}^d) \cap L^2_k(\mathbb{R}^d)$ when $s > \frac{d+2\gamma}{2}$. Thus the function $\mathfrak{K}_s$ is well defined and we can write

$$\mathfrak{K}_s(x, y) = (\mathscr{F}_D)^{-1}(\Upsilon_y)(x), \quad \text{for all } x \in \mathbb{R}^d.$$

Moreover, from Proposition 2.2, we can see that the function $\mathfrak{K}_s(., y)$ belongs to $L^2_k(\mathbb{R}^d)$, and we have

$$\mathscr{F}_D\Big(\mathfrak{K}_s(., y)\Big)(\xi) = \frac{K(-i\xi, y)}{(1+||\xi||^2)^s}. \tag{7.2}$$

As $K(-i\xi, y)$ is bounded, we obtain

$$|\mathscr{F}_D\Big(\mathfrak{K}_s(., y)\Big)(\xi)| \leq \frac{1}{(1+||\xi||^2)^s}$$

and

$$||\mathfrak{K}_s(., y)||^2_{W^s_k(\mathbb{R}^d)} \leq C(s) := \int_{\mathbb{R}^d} \frac{d\gamma_k(\xi)}{(1+||\xi||^2)^s} < \infty. \tag{7.3}$$

This proves that for all $y \in \mathbb{R}^d$ the function $\mathfrak{K}_s(., y)$ belongs to $W^s_k(\mathbb{R}^d)$.

(ii) Let f be in $W^s_k(\mathbb{R}^d)$ and y in $\mathbb{R}^d$. From (7.1) and (7.2), we have

$$\langle f, \mathfrak{K}_s(., y)\rangle_{W^s_k(\mathbb{R}^d)} = \int_{\mathbb{R}^d} \mathscr{F}_D(f)(\xi)\overline{K(-i\xi, y)} d\gamma_k(\xi), \tag{7.4}$$

and from inversion formula, we obtain the reproducing property

$$f(y) = \langle f, \mathfrak{K}_s(x, y)\rangle_{W^s_k(\mathbb{R}^d)}.$$

This completes the proof of the proposition. □

Corollary 7.1. *For $s > \frac{d+2\gamma}{2}$, the generalized Sobolev space $W^s_k(\mathbb{R}^d)$ is embedded in $C(\mathbb{R}^d)$.*

7.1. Extremal functions associated with the partial Dunkl wavelet transform.
Let h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$ and let $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$. We denote by $S^D_{h,b}$ the partial Dunkl wavelet transform defined by

$$S^D_{h,b}(f) := \mathscr{N}_h^D(f)(b,.), \quad \text{for all } f \in L^2_k(\mathbb{R}^d).$$

Proposition 7.2. *Let h be a Dunkl wavelet in $L^1_k(\mathbb{R}^d)\bigcap L^2_k(\mathbb{R}^d)$ and let b belongs to $\mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$. The transformation $S^D_{h,b}$ is a bounded linear operator from $W^s_k(\mathbb{R}^d)$, $s \geq 0$, into $L^2_k(\mathbb{R}^d)$, and there exists a positive constant $C_k(b,h)$ such that we have*

$$\|S^D_{h,b}f\|_{L^2_k(\mathbb{R}^d)} \leq C_k(b,h)\|f\|_{W^s_k(\mathbb{R}^d)}, \quad f \in W^s_k(\mathbb{R}^d).$$

Proof. Using the relation (3.14), Proposition 2.8, (3.17) and Plancherel's formula (2.15) we obtain the result. □

Let $r > 0$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$, $s \geq 0$ and h be a Dunkl wavelet wavelet in $L^1_k(\mathbb{R}^d)\bigcap L^2_k(\mathbb{R}^d)$. We introduce the inner product in the space $W^s_k(\mathbb{R}^d)$

$$\langle f,g\rangle_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)} = r\langle f,g\rangle_{W^s_k(\mathbb{R}^d)} + \langle S^D_{h,b}(f), S^D_{h,b}(g)\rangle_{L^2_k(\mathbb{R}^d)}, \quad f,g \in W^s_k(\mathbb{R}^d).$$

The norm associated to the inner product is defined by:

$$\|f\|^2_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)} := r\|f\|^2_{W^s_k(\mathbb{R}^d)} + \|S^D_{h,b}(f)\|^2_{L^2_k(\mathbb{R}^d)}.$$

Remark 7.1. *Simple calculations give that $||.||_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)}$ and $||.||_{W^s_k(\mathbb{R}^d)}$ are equivalent for $r > 0$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$.*

Proposition 7.3. *Let $r > 0$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$, $s > \frac{d+2\gamma}{2}$ and h be a Dunkl wavelet in $L^1_k(\mathbb{R}^d)\bigcap L^2_k(\mathbb{R}^d)$. Then the generalized Sobolev space $(W^s_k(\mathbb{R}^d), \langle .,.\rangle_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)})$, possesses a reproducing kernel $\mathscr{K}_{S^D_{h,b},r}$ satisfying the identity*

$$\mathscr{K}_{S^D_{h,b},r}(.,y) = \Big(rI + (S^D_{h,b})^*S^D_{h,b}\Big)^{-1}\mathfrak{K}_s(.,y) \tag{7.5}$$

where $(S^D_{h,b})^* : L^2_k(\mathbb{R}^d) \longrightarrow W^s_k(\mathbb{R}^d)$ is the adjoint operator of $S^D_{h,b}$ given by

$$\langle S^D_{h,b}(f), g\rangle_{L^2_k(\mathbb{R}^d)} = \langle f, (S^D_{h,b})^*g\rangle_{W^s_k(\mathbb{R}^d)}, \quad f \in W^s_k(\mathbb{R}^d),\ g \in L^2_k(\mathbb{R}^d).$$

Moreover the kernel $\mathscr{K}_{S^D_{h,b},r}$ satisfies the following properties:

(i) $||\mathscr{K}_{S^D_{h,b},r}(.,y)||_{W^s_k(\mathbb{R}^d)} \leq \frac{C(s)}{r}$, for all $y \in \mathbb{R}^d$,

(ii) $||S^D_{h,b}\big(\mathscr{K}_{S^D_{h,b},r}(.,y)\big)||_{L^2_k(\mathbb{R}^d)} \leq \frac{C(s)}{\sqrt{r}}$, for all $y \in \mathbb{R}^d$,

(iii) $||(S^D_{h,b})^*S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(.,y))||_{W^s_k(\mathbb{R}^d)} \leq C(s)$, for all $y \in \mathbb{R}^d$. Here $C(s)$ is the constant given by (7.3).

Proof. From Corollary 7.1, Proposition 7.2 and Remark 7.1, we deduce that the map $u \mapsto u(y)$, $y \in \mathbb{R}^d$, is a continuous linear functional on the space

$(W^s_k(\mathbb{R}^d), \langle ., . \rangle_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)})$. Thus from [27], $(W^s_k(\mathbb{R}^d), \langle ., . \rangle_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)})$ has a reproducing kernel denoted by $\mathscr{K}_{S^D_{h,b},r}$. On the other hand, we have

$$\begin{aligned} f(y) &= \langle f, \mathscr{K}_{S^D_{h,b},r}(., y)\rangle_{S^D_{h,b},r,W^s_k(\mathbb{R}^d)}. \\ &= r\langle f, \mathscr{K}_{S^D_{h,b},r}(., y)\rangle_{W^s_k(\mathbb{R}^d)} + \langle S^D_{h,b}(f), S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(., y))\rangle_{L^2_k(\mathbb{R}^d)} \\ &= \langle f, \Big(rI + (S^D_{h,b})^* S^D_{h,b}\Big)\mathscr{K}_{S^D_{h,b},r}(., y)\rangle_{W^s_k(\mathbb{R}^d)}. \end{aligned}$$

Thus,

$$(rI + (S^D_{h,b})^* S^D_{h,b})\mathscr{K}_{S^D_{h,b},r}(., y) = \mathfrak{K}_s(., y). \tag{7.6}$$

Furthermore, the previous identity implies that

$$\begin{aligned} &r^2||\mathscr{K}_{S^D_{h,b},r}(., y)||^2_{W^s_k(\mathbb{R}^d)} + 2r||S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(., y))||^2_{L^2_k(\mathbb{R}^d)} + \\ &||(S^D_{h,b})^* S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(., y))||^2_{W^s_k(\mathbb{R}^d)} \\ &= ||\mathfrak{K}_s(., y)||^2_{W^s_k(\mathbb{R}^d)}. \end{aligned}$$

From this relation and using the fact that

$$||\mathfrak{K}_s(., y)||_{W^s_k(\mathbb{R}^d)} \leq C(s),$$

we obtain the properties (i), (ii) and (iii). □

Remark 7.2. *Using a similar ideas as in Proposition 7.1, we prove that*

$$\mathscr{K}_{S^D_{h,b},r}(x, y) = \int_{\mathbb{R}^d} \frac{K(-i\xi, x)K(i\xi, y)}{r(1 + ||\xi||^2)^s + |\mathscr{F}_D(\Delta_b h)(\xi)|^2} d\gamma_k(\xi).$$

We can now state the main result of this paragraph.

Theorem 7.1. *Let $s > \frac{d+2\gamma}{2}$ and h be a Dunkl wavelet in $L^1_k(\mathbb{R}^d) \bigcap L^2_k(\mathbb{R}^d)$.*

(i) For any $g \in L^2_k(\mathbb{R}^d)$ and for any $r > 0$, $b \in \mathbb{R}^d \backslash \{0_{\mathbb{R}^d}\}$, the best approximate function $f^*_{r,b,g}$ in the sense

$$\inf_{f \in W^s_k(\mathbb{R}^d)} \Big\{ r\|f\|^2_{W^s_k(\mathbb{R}^d)} + \|g - S^D_{h,b}(f)\|^2_{L^2_k(\mathbb{R}^d)} \Big\} \tag{7.7}$$

exists uniquely and it is represented by

$$f^*_{r,b,g}(y) = \langle g, S^D_{h,b}\big(\mathscr{K}_{S^D_{h,b},r}(., y)\big)\rangle_{L^2_k(\mathbb{R}^d)}. \tag{7.8}$$

(ii) The extremal function $f^*_{r,b,g}$ satisfies the following inequality:

$$|f^*_{r,b,g}(y)| \leq \frac{C(s)}{\sqrt{r}}||g||_{L^2_k(\mathbb{R}^d)}.$$

Proof. (i) The existence and uniqueness of extremal function $f^*_{r,b,g}$ satisfying (7.7) is given by [28], and the extremal function $f^*_{r,b,g}$ is represented by

$$f^*_{r,b,g}(y) = \langle g, S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(., y))\rangle_{L^2_k(\mathbb{R}^d)}, \quad y \in \mathbb{R}^d.$$

(ii) From Proposition 7.3 (ii), we have

$$|f^*_{r,b,g}(y)| \leq ||g||_{L^2_k(\mathbb{R}^d)}||S^D_{h,b}(\mathscr{K}_{S^D_{h,b},r}(., y))||_{L^2_k(\mathbb{R}^d)} \leq \frac{C(s)}{\sqrt{r}}||g||_{L^2_k(\mathbb{R}^d)}.$$

Thus the theorem is proved. □

Corollary 7.2. *Let $s > \frac{d+2\gamma}{2}$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$ and $r > 0$. If f is in $W_k^s(\mathbb{R}^d)$ and $g = S_{h,b}^D(f)$. Then*

(i) $f(y) = \lim_{r\to 0^+} f_{r,b,g}^*(y)$, for all $y \in \mathbb{R}^d$.

(ii) $|f(y) - f_{r,b,g}^*(y)| \leq C(s)||f||_{W_k^s(\mathbb{R}^d)}$, for all $y \in \mathbb{R}^d$.

(iii) $|f_{r,b,g}^*(y)| \leq C(s)||f||_{W_k^s(\mathbb{R}^d)}$, for all $y \in \mathbb{R}^d$, where $C(s)$ is the constant given by (7.3).

Proof. Let f be in $W_k^s(\mathbb{R}^d)$.

(i) Then

$$f_{r,b,g}^*(y) = \langle f, (S_{h,b}^D)^* S_{h,b}^D(\mathscr{K}_{S_{h,b}^D,r}(.,y))\rangle_{W_k^s(\mathbb{R}^d)}. \tag{7.9}$$

But from (7.6), we have

$$\lim_{r\to 0^+}(S_{h,b}^D)^* S_{h,b}^D(\mathscr{K}_{S_{h,b}^D,r}(.,y)) = \mathfrak{K}_s(.,y).$$

Thus

$$\lim_{r\to 0^+} f_{r,b,g}^*(y) = \langle f, \mathfrak{K}_s(.,y)\rangle_{W_k^s(\mathbb{R}^d)} = f(y).$$

(ii) From (7.6) and (7.9), the extremal function $f_{r,b,g}^*$ satisfies

$$f_{r,b,g}^*(y) = f(y) - r\langle f, \mathscr{K}_{S_{h,b}^D,r}(.,y)\rangle_{W_k^s(\mathbb{R}^d)}.$$

Thus and by Proposition 7.3 (i) we obtain

$$|f_{r,b,g}^*(y) - f(y)| \leq r||f||_{W_k^s(\mathbb{R}^d)}||\mathscr{K}_{S_{h,b}^D,r}(.,y)||_{W_k^s(\mathbb{R}^d)} \leq C(s)||f||_{W_k^s(\mathbb{R}^d)}.$$

(iii) From (7.9) and Proposition 7.3 (iii), the extremal function $f_{r,b,g}^*$ satisfies

$$|f_{r,b,g}^*(y)| \leq ||f||_{W_k^s(\mathbb{R}^d)}||(S_{h,b}^D)^* S_{h,b}^D(\mathscr{K}_{S_{h,b}^D,r}(.,y))||_{W_k^s(\mathbb{R}^d)} \leq C(s)||f||_{W_k^s(\mathbb{R}^d)}.$$

□

Remark 7.3. *Let $s > \frac{d+2\gamma}{2}$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$ and $r > 0$. If $S_{h,b}^D$ is isometry (i.e. $(S_{h,b}^D)^* S_{h,b}^D = Id$) then*

(i) $\langle .,.\rangle_{S_{h,b}^D,r,W_k^s(\mathbb{R}^d)} = (r+1)\langle .,.\rangle_{W_k^s(\mathbb{R}^d)}$.

(ii) $\mathscr{K}_{S_{h,b}^D,r}(x,y) = \frac{1}{r+1}K_s(x,y)$, for all $x, y \in \mathbb{R}^d$.

(iii) For all $y \in \mathbb{R}^d$, $f_{r,b,g}^*(y) = \frac{1}{r+1}(S_{h,b}^D)^* g(y)$, $g \in L_k^2(\mathbb{R}^d)$.

(iv) For all $y \in \mathbb{R}^d$, $f_{r,b,S_{h,b}^D(u)}^*(y) = \frac{1}{r+1}u(y)$, $u \in W_k^s(\mathbb{R}^d)$.

Using similar ideas as in [19], we obtain the following results:

Proposition 7.4. *Let $s > \frac{d+2\gamma}{2}$ and h be a Dunkl wavelet in $L_k^1(\mathbb{R}^d)\bigcap L_k^2(\mathbb{R}^d)$.*

i*) For any $g \in L_k^2(\mathbb{R}^d)$, $b \in \mathbb{R}^d\backslash\{0_{\mathbb{R}^d}\}$ and for any $r > 0$, the best approximate function $f_{r,b,g}^*$ is represented by*

$$f_{r,b,g}^*(x) = \int_{\mathbb{R}^d} g(y)Q_{r,b,g}(x,y)d\gamma_k(y),$$

where

$$Q_{r,b,g}(x,y) = \int_{\mathbb{R}^d} \frac{\mathscr{F}_D(\Delta_b h)(\xi)K(-i\xi,x)K(i\xi,y)}{r(1+||\xi||^2)^s + |\mathscr{F}_D(\Delta_b h)(\xi)|^2}d\gamma_k(\xi).$$

ii*) If we take* $g = S^D_{h,b}(f)$, *then*

$$\lim_{r\to 0^+} ||f^*_{r,b,g} - f||_{W^s_k(\mathbb{R}^d)} = 0.$$

Moreover, $\{f^*_{r,b,g}\}_{r>0}$ *converges uniformly to* f *as* $r \to 0^+$.

iii*) Let* $\delta > 0$ *and let* g, g_δ *satisfy* $\|g - g_\delta\|_{L^2_k(\mathbb{R}^d)} \leq \delta$. *Then*

$$\|f^*_{r,b,g} - f^*_{r,b,g_\delta}\|_{W^s_k(\mathbb{R}^d)} \leq \frac{\delta}{\sqrt{r}}.$$

7.2. The extremal function associated with the Dunkl wavelet transform.

Let $r > 0$, $s \geq 0$ and h be a Dunkl wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$. We introduce the inner product in the space $W^s_k(\mathbb{R}^d)$

$$\langle f, g\rangle_{\mathscr{N}^D_h, r, W^s_k(\mathbb{R}^d)} = r\langle f, g\rangle_{W^s_k(\mathbb{R}^d)} + \langle \mathscr{N}^D_h(f), \mathscr{N}^D_h(g)\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})}, \quad f, g \in W^s_k(\mathbb{R}^d).$$

The norm associated to the inner product is define by:

$$\|f\|^2_{\mathscr{N}^D_h, r, W^s_k(\mathbb{R}^d)} := r\|f\|^2_{W^s_k(\mathbb{R}^d)} + \|\mathscr{N}^D_h(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{2d})}.$$

From (3.19), the inner product $\langle .,, \rangle_{\mathscr{N}^D_h, r, W^s_k(\mathbb{R}^d)}$ can be written as

$$\langle f, g\rangle_{\mathscr{N}^D_h, r, W^s_k(\mathbb{R}^d)} = r\langle f, g\rangle_{W^s_k(\mathbb{R}^d)} + C_h\langle f, g\rangle_{L^2_k(\mathbb{R}^d)}. \tag{7.10}$$

Using similar ideas as in Proposition 7.3, Theorem 7.1 and Corollary 7.2, we obtain the following results:

Proposition 7.5. *Let* $s > \frac{d+2\gamma}{2}$ *and* h *be a generalized wavelet on* $\mathbb{R}^d$ *in* $L^2_k(\mathbb{R}^d)$. *Then the generalized Sobolev space* $(W^s_k(\mathbb{R}^d), \langle ., .\rangle_{\mathscr{N}^D_h, r, W^s_k(\mathbb{R}^d)})$, *admits the following reproducing kernel*

$$\mathfrak{K}^D_{r,h}(x, y) = \int_{\mathbb{R}^d} \frac{K(i\xi, x)K(-i\xi, y)}{r(1 + ||\xi||^2)^s + C_h} d\gamma_k(\xi).$$

Moreover the kernel $\mathfrak{K}^D_{r,h}$ *satisfies the following properties:*

(i) $\mathfrak{K}^D_{r,h}(x, y) = \Big(rI + (\mathscr{N}^D_h)^*\mathscr{N}^D_h\Big)^{-1}\mathfrak{K}_s(., y)$,
where

$$(\mathscr{N}^D_h)^* : L^2_{\mu_k}(\mathbb{R}^{2d}) \longrightarrow W^s_k(\mathbb{R}^d)$$

is the adjoint operator of $\mathscr{N}^D_h$ *given by*

$$\langle \mathscr{N}^D_h(f), h\rangle_{L^2_{\mu_k}(\mathbb{R}^{2d})} = \langle f, (\mathscr{N}^D_h)^* h\rangle_{W^s_k(\mathbb{R}^d)}, \quad f \in W^s_k(\mathbb{R}^d),\ h \in L^2_{\mu_k}(\mathbb{R}^{2d}).$$

(ii) $||\mathfrak{K}^D_{r,h}(., y)||_{W^s_k(\mathbb{R}^d)} \leq \frac{C(s)}{r}$, *for all* $y \in \mathbb{R}^d$.

(iii) $||\mathscr{N}^D_h\mathfrak{K}^D_{r,h}(., y)||_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \frac{C(s)}{\sqrt{r}}$, *for all* $y \in \mathbb{R}^d$.

(iv) $||(\mathscr{N}^D_h)^*\mathscr{N}^D_h\big(\mathfrak{K}^D_{r,h}(., y)\big)||_{W^s_k(\mathbb{R}^d)} \leq C(s)$, *for all* $y \in \mathbb{R}^d$, *where* $C(s)$ *is the constant given by* (7.3).

Theorem 7.2. *Let $s > \frac{d+2\gamma}{2}$ and h be a generalized wavelet on $\mathbb{R}^d$ in $L^2_k(\mathbb{R}^d)$.*

i) *For any $g \in L^2_{\mu_k}(\mathbb{R}^{2d})$ and for any $r > 0$, the best approximate function $f^*_{r,g}$ in the sense*

$$\inf_{f\in W^s_k(\mathbb{R}^d)}\Big\{r\|f\|^2_{W^s_k(\mathbb{R}^d)} + \|g - \mathscr{N}_h^D(f)\|^2_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}\Big\}$$
$$= r\|f^*_{r,g}\|^2_{W^s_k(\mathbb{R}^d)} + \|g - \mathscr{N}_h^D(f^*_{r,g})\|^2_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)}$$

exists uniquely and $f^*_{r,g}$ is represented by

$$f^*_{r,g}(x) = \langle g, \mathscr{N}_h^D\big(\mathfrak{K}^D_{r,h}(.,x)\big)\rangle_{L^2_{\mu_k}(\mathbb{R}^{d+1}_+)} = \int_{\mathbb{R}^{d+1}_+} g(b,y)Q^D_{r,h}(b,x,y)d\mu_k(b,y),$$

where

$$Q^D_{r,h}(b,x,y) = \int_{\mathbb{R}^d} \frac{\mathscr{F}_D(\Delta_b h)(\xi)K(-i\xi,x)K(i\xi,y)}{r(1+||\xi||^2)^s + C_h}d\gamma_k(\xi).$$

ii) *If we take $g = \mathscr{N}_h^D(f)$, then*

$$\lim_{r\to 0^+} ||f^*_{r,g} - f||_{W^s_k(\mathbb{R}^d)} = 0.$$

Moreover, $\{f^*_{r,g}\}_{r>0}$ converges uniformly to f as $r \to 0^+$.

iii) *Let $\delta > 0$ and let g, g_δ satisfy $\|g - g_\delta\|_{L^2_{\mu_k}(\mathbb{R}^{2d})} \leq \delta$. Then*

$$\|f^*_{r,g} - f^*_{r,g_\delta}\|_{W^s_k(\mathbb{R}^d)} \leq \frac{\delta}{\sqrt{r}}.$$

(iv) $f(y) = \lim_{r\to 0^+} f^*_{r,g}(y)$, for all $y \in \mathbb{R}^d$.
(v) $|f(y) - f^*_{r,g}(y)| \leq C(s)||f||_{W^s_k(\mathbb{R}^d)}$, for all $y \in \mathbb{R}^d$.
(vi) $|f^*_{r,g}(y)| \leq C(s)||f||_{W^s_k(\mathbb{R}^d)}$, for all $y \in \mathbb{R}^d$.

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References

[1] L. D. Abreu, K. Grchenig and J. L. Romero, On accumulated spectrograms. Trans. Amer. Math. Soc. 368(2016): 3629–3649.

[2] F. Bouzeffour, On the norm of the L^p Dunkl transform. Applicable Analysis 94(4)(2015): 761-779.

[3] L. Debnath and F.A. Shah, Wavelet Transforms and Their Applications. Birkhäuser, 2015.

[4] M.F.E. de Jeu, The Dunkl transform. Invent.Math. 113(1993): 147-162.

[5] D. L. Donoho and P. B. Stark, Uncertainty principles and signal recovery. SIAM J. Appl. Math. 49(1989): 906-931.

[6] C.F. Dunkl, *Differential-difference operators associated to reflection groups. Trans. Am. Math. Soc. 311(1989): 167-183.*

[7] C.F. Dunkl, Hankel transforms associated to finite reflection groups. Contemp. Math. 138(1992): 123-138.

[8] D. Gabor, Theory of communication. part 1: The analysis of information. Journal of the Institution of Electrical Engineers-Part III: Radio and Communication Engineering 93(26)(1946): 429-441.

[9] L. Gallardo and K. Trimèche, An L^p version of Hardy's theorem for the Dunkl transform. Journal of the Australian Mathematical Society 77(03)(2004): 371-386.

[10] S. Ghobber and P. Jaming, Uncertainty principles for integral orperators. Studia Math. 220(2014): 197-220.
[11] V. Havin and B. Jöricke, The uncertainty principle in harmonic analysis. Volume 24. Berlin: Springer Verlag, 1994.
[12] T. Kawazoe, and H. Mejjaoli, Uncertainty principles for the Dunkl transform. Hiroshima Math. J. 40(2)(2010): 241-268.
[13] E. Malinnikova, Orthonormal sequences in $L^2(\mathbb{R}^d)$ and time frequency localization. J. Fourier Anal. Appl. 16(6)(2010): 983-1006.
[14] H. Mejjaoli, K. Trimèche, Hypoellipticity and hypoanalyticity of the Dunkl Laplacian operator. Integ. Transf. and Special Funct. 15(6)(2004): 523-548.
[15] H. Mejjaoli and K. Trimèche, Spectrum of functions for the Dunkl transform on $\mathbb{R}^d$, Fractional Calculus and Applied Analysis 10(1)(2007): 19-38.
[16] H. Mejjaoli, N. Sraieb, Uncertainty principles for the continuous Dunkl wavelet transform and the Dunkl continuous Gabor transform. Mediterr. J. Math. 5(2008): 443-466.
[17] H. Mejjaoli, Strichartz estimates for the Dunkl wave equation and application. J. Math. Anal. Appl. 346(2008): 41-54.
[18] H. Mejjaoli, Dispersion phenomena in Dunkl-Schrödinger equation and applications. Serdica Math. J. 35(1)(2009): 25-60.
[19] H. Mejjaoli, Practical inversion formulas for the Dunkl-Gabor transform on $\mathbb{R}^d$. Integ. Transf. and Special Funct. 23(12)(2012): 875-890.
[20] Dunkl-Schrödinger semigroups and applications. Applicable Analysis 92(8)(2013): 1597-1626.
[21] H. Mejjaoli and K. Trimèche, Time-frequency concentration, Heisenberg type uncertainty principles and localization operators for the continuous Dunkl wavelet transform on $\mathbb{R}^d$. Mediterranean Journal of Mathematics 14(146)(2017).
[22] H. Mejjaoli, Dunkl-Stockwell transform and its applications to the time-frequency analysis. Journal of Pseudo-Differential Operators and Applications 12(2)(2021): 1-59.
[23] F. Riesz and B. Sz.-Nagy, Functional Analysis (Frederick Ungar Publishing Co., New York, 1995).
[24] M. Rösler and M. Voit, Markov processes related with Dunkl operators. Adv. in Appl. Math. 21(4)(1998): 575-643.
[25] M. Rösler, Positivity of Dunkl's intertwining operator. Duke Math. J. 98(1999): 445-463.
[26] M. Rösler, A positive radial product formula for the Dunkl kernel. Trans. Am. Math. Soc. 355(2003): 24132438.
[27] S. Saitoh, Theory of reproducing kernels and its applications. Longman Scientific Technical, Harlow, 1988.
[28] S. Saitoh, Reproducing Kernels and their Applications. Pitman Research Notes in Math. Series 369, Addison Wesley Longman, Harlow, 1997.
[29] S. Thangavelu and Y. Xu, Convolution operator and maximal functions for Dunkl transform. J. d'Analyse Mathematique 97(2005): 25-56.
[30] S. Thangavelu and Y. Xu, Riesz transform and Riesz potentials for Dunkl transform, Journal of Computational and Applied Mathematics 199(1)(2007): 181-195.
[31] K. Trimèche, Paley-Wiener theorems for Dunkl transform and Dunkl translation operators.Integ. Transf. and Special Funct. 13(2002): 17-38.
[32] K. Trimèche, Inversion of the Dunkl intertwining operator and its dual using Dunkl wavelet. Rocky Mountain Journal of Mathematics 32(2)(2002): 889-918.
[33] K. Trimèche, Generalized Wavelets and Hypergroups. Gordon and Breach Science Publishers, 1997.
[34] M.W. Wong, Wavelet transforms and localization operators. 136, Springer Science & Business Media, 2002.

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