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GEOMETRY AND DIVISION BY ZERO CALCULUS

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Abstract. We demonstrate several results in plane geometry derived from division by zero and division by zero calculus. The results show that the two new concepts open an entirely new world of mathematics.

1. Introduction

The lack of division by zero has been a serious glaring omission in our mathematics. While recent publications [3], [7], [8], [22], [29] have offered a final solution to this [3]:

$$\frac{z}{0} = 0 \quad for\ any\ element\ z\ in\ a\ field. \tag{1.1}$$

In this article we show several results in plane geometry obtained from division by zero given by (1.1) and division by zero calculus, which is a generalization of division by zero.

For a meromorphic function $W = f(z)$, we consider the Laurent expansion of f around $z = a$:

$$W = f(z) = \sum_{n=-\infty}^{n=-1} C_n(z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n(z-a)^n.$$

Then we *define* $f(a) = C_0$. This is a generalization of (1.1) called *division by zero calculus* [29]. Now we can consider the value $f(a) = C_0$ at an isolated singular point a.

We consider some families of circles in the plane, each of the members is represented by a Cartesian equation $f_z(x, y) = 0$ with parameter $z \in \mathbb{R}$. Here, we assume that if x and y are fixed, $f_z(x, y)$ is a meromorphic function in z. Then, for the Laurent expansion of the function $f_z(x, y)$ at $z = a$ for fixed x, y, the corresponding coefficient $C_n(a; x, y)$ is depending on also x and y.

In this setting we will see some mysterious relation with the equation

$$f_z(x, y) = 0$$

and the equations

$$C_n(a; x, y) = 0,$$

for fixed a. However we will see that the equation $C_n(a; x, y) = 0$ implies some meaningful things even for an integer $n \neq 0$ also in the case in which division

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by zero does not occur for $z = a$. Moreover we will show that the equation $C_n(a; x, y) = 0$ gives some notable and meaningful figures, which have never been considered before. On the other hand, we have no idea why the coefficients of the Laurent expansion show such meaningful and marvelous facts at the present time of writing. Therefore we can only show such results with little explanations in this paper.

For lines and circles, (1.1) gives a totally new insight, which are essential to our paper. The results are stated as follows:

Proposition 1.1 ([8],[29]). *The following statements are true.*
(i) *We can regard a line as a circle with its radius* 0*, when we consider a line as a special case of a circle.*
(ii) *We can consider that orthogonal figures touch to each other, in a natural interpretation.*

Proof. Any circle in the plane has an equation

$$e(x^2 + y^2) + 2fx + 2gy + h = 0,$$

and has radius

$$\sqrt{\frac{f^2 + g^2 - eh}{e^2}}. \tag{1.2}$$

While the equation represents a line in the case $e = 0$, and (1.2) equals 0 in this case by (1.1). This proves (i). For tangential figures, their angle θ of two tangential lines at the common point is zero and $\tan(\theta) = 0$. However $\tan(\pi/2) = 0$, by (1.1) and by the division by zero calculus and so in this sense, we can say (ii). □

2. Triangle with parallel sides

Let us consider a triangle ABC in the plane with $a = |BC|$, $b = |CA|$ and $c = |AB|$. Let θ_a (resp. θ_b) be the angle between $\overrightarrow{BA}$ and $\overrightarrow{AC}$ (resp. $\overrightarrow{BC}$) (see Figure 1). We fix the points A and B and the angle θ_b, and consider the side length, the circumradius and so forth of the triangle ABC in the case $\theta_a = \theta_b$ by the definition of division by zero (1.1) (see Figure 2). We use a rectangular coordinate system such that A and B have coordinates $(p, 0)$ and $(q, 0)$, respectively such that $p - q = c$, where we assume that the point C lies on the region $y > 0$.

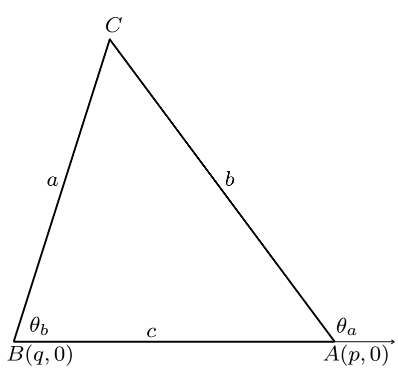

Figure 1.

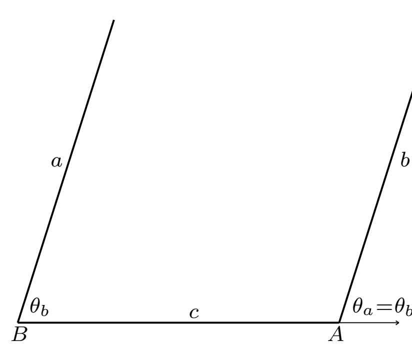

Figure 2.

2.1. Side length and area. The lines AC and BC have equations $y\cos\theta_a = (x-p)\sin\theta_a$ and $y\cos\theta_b = (x-q)\sin\theta_b$, respectively. Therefore the point C, which is the point of intersection of the two line, has coordinates

$$(x_c, y_c) = \left(\frac{p\sin\theta_a\cos\theta_b - q\cos\theta_a\sin\theta_b}{\sin(\theta_a-\theta_b)}, \frac{c\sin\theta_a\sin\theta_b}{\sin(\theta_a-\theta_b)}\right). \tag{2.1}$$

Therefore from $a = \sqrt{(x_c-q)^2+y_c^2}$ and $b = \sqrt{(x_c-p)^2+y_c^2}$, we get

$$a = \frac{c\sin\theta_a}{\sin(\theta_a-\theta_b)}, \quad b = \frac{c\sin\theta_b}{\sin(\theta_a-\theta_b)}. \tag{2.2}$$

If $\theta_a = \theta_b$, then $\sin(\theta_a-\theta_b) = 0$, and we get $a = b = 0$ by (1.1). The y-coordinate in (2.1) shows that the height corresponding to the base AB equals 0 if $\theta_a = \theta_b$. Therefore we have:

Theorem 2.1. *The side length of the parallel sides of a triangle equals 0. Also the area of a triangle with parallel sides equals 0.*

Also (2.1) shows that the point C coincides with the origin $(0,0)$ if $\theta_a = \theta_b$.

2.2. Circumradius. Let R be the circumradius of the triangle ABC.

Theorem 2.2. *The circumradius of a triangle with parallel sides equals 0.*

Proof. We use the identity

$$R = \frac{abc}{\sqrt{(a+b+c)(-a+b+c)(a-b+c)(a+b-c)}}.$$

Substituting (2.2), in the above equation, we have

$$R = \frac{c}{2\sin(\theta_a-\theta_b)}.$$

Therefore we get $R = 0$ if $\theta_a = \theta_b$. □

Let r be the inradius of ABC. We consider the following identity:

$$R = \frac{r}{\cos A + \cos B + \cos C - 1}.$$

The identity is true in the case $\theta_a = \theta_b$, because the left side equals $R = 0$ by Theorem 2.2, and the denominator of the right side equals $\cos A + \cos B + \cos C - 1 = \cos(\pi-\theta_b) + \cos\theta_b + \cos 0 - 1 = 0$. Therefore the right side also equals 0 by (1.1).

2.3. Excircle. We consider the excircle of the triangle ABC touching CA from the side opposite to B (see Figure 3).

Theorem 2.3 ([11])**.** *If $\theta_a = \theta_b$, then the radius of the excircle of the triangle ABC touching CA from the side opposite to B equals* 0.

Proof. The center of the excircle coincides with the point of intersection of the lines represented by $y = \tan\frac{\theta_a}{2}(x-p)$ and $y = \tan\frac{\theta_b}{2}(x-q)$, and has coordinates

$$\left(\frac{p\tan\frac{\theta_a}{2} - q\tan\frac{\theta_b}{2}}{\tan\frac{\theta_a}{2} - \tan\frac{\theta_b}{2}}, c\frac{\sin\frac{\theta_a}{2}\sin\frac{\theta_b}{2}}{\sin\frac{\theta_a-\theta_b}{2}}\right),$$

where the y-coordinate gives the exradius. While the y-coordinate equals 0 if $\theta_a = \theta_b$. The proof is complete. $\square$

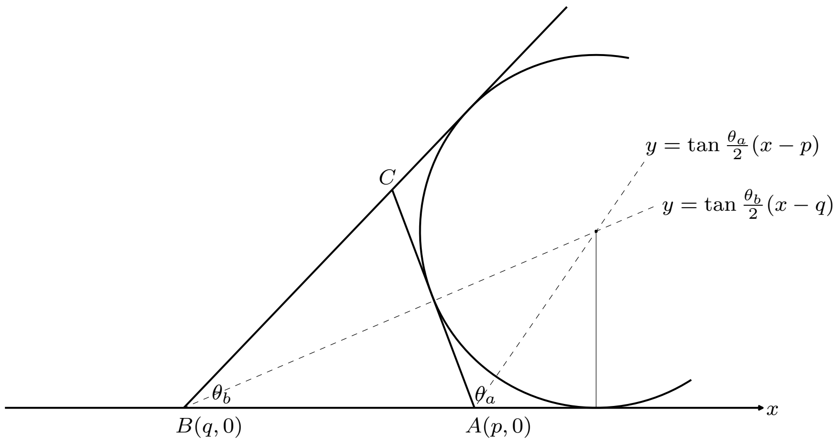

Figure 3.

Notice that the center of the excircle coincides with the origin if $\theta_a = \theta_b$.

Remark 2.4. The essential fact of this section is that the point of intersection of two parallel lines coincides with the origin [8], [29].

3. Pompe's theorem

Generalizing a problem in Wasan geometry (Japanese old geometry), W. Pompe gives the following theorem [6] (see Figure 4):

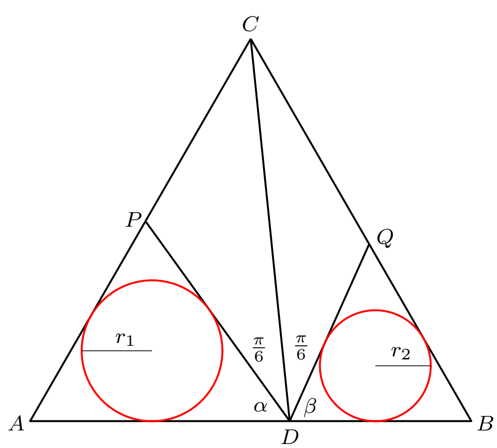

Figure 4.

Theorem 3.1 ([28])**.** *For an equilateral triangle ABC, let D be a point on the side AB. For points P and Q lying on the sides AC and BC, respectively, satisfying*

$\angle PDC = \angle QDC = \pi/6$, let $\alpha = \angle ADP$ and $\beta = \angle BDQ$. If r_1 and r_2 are the inradii of the triangles ADP and BDQ, respectively, then we have

$$\frac{r_1}{r_2} = \frac{\sin 2\alpha}{\sin 2\beta}. \tag{3.1}$$

In this section we consider the case $\beta = \pi/2$ in the sense of division by zero and division by zero calculus. In this case the point D coincides with B, then the triangle BQD degenerates to the point B, i.e., $r_2 = 0$ (see Figure 5). In this case the left side of (3.1) equals $r_1/0 = 0$. Also the right side equals $\sin 2\alpha/\sin 2\pi = \sin 2\alpha/0 = 0$. Therefore (3.1) holds by (1.1).

On the other hand the right side of (3.1) is a function of β; $\sin 2(2\pi/3 - \beta)/\sin 2\beta$. By the Laurent expansion of this about $\beta = \pi/2$:

$$\frac{\sin 2(2\pi/3 - \beta)}{\sin 2\beta} = \cdots - \frac{\sqrt{3}}{4}\left(\beta - \frac{\pi}{2}\right)^{-1} + \frac{1}{2} + \frac{1}{\sqrt{3}}\left(\beta - \frac{\pi}{2}\right) + \cdots,$$ [1]

we get

$$\frac{r_1}{r_2} = \frac{\sin 2\alpha}{\sin 2\beta} = \frac{1}{2}$$

in the case $\beta = \pi/2$. The large circle in Figure 6 has radius $r_2 = 2r_1$ and center $B = Q$. It is orthogonal to the lines AB, BC and the perpendicular to AB at B. Therefore the circle still touches the three lines by Proposition 1.1, i.e., *it is the circle of radius $2r_1$ touching the lines AB, BC and the perpendicular to AB at B.*

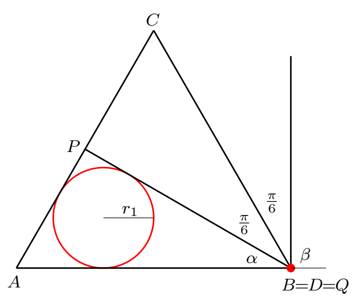

Figure 5.

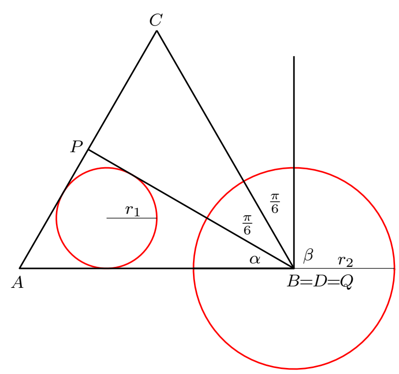

Figure 6.

4. A circle touching a circle and its tangent

For a circle α of radius a, let O be a point lying on α. We use a rectangular coordinate system with origin O such that the center of α has coordinates $(0, a)$.

Let β be a fixed circle of radius b touching α and the x-axis in the first quadrant for a positive real number b (see Figure 7). Let γ be another circle of radius r touching α and the x-axis in the first quadrant. We consider the case γ has radius b by division by zero calculus. The circle γ is represented by the equation

$$\gamma(x, y) = (x - 2\sqrt{ar})^2 + (y - r)^2 - r^2 = 0. \tag{4.1}$$

[1]There are typos in the expansion in [6].

We now consider the Laurent expansion of $\gamma(x, y)$ about $r = b$:

$$\gamma(x, y) = \sum_{n=-\infty}^{\infty} C_n (r - b)^n.$$

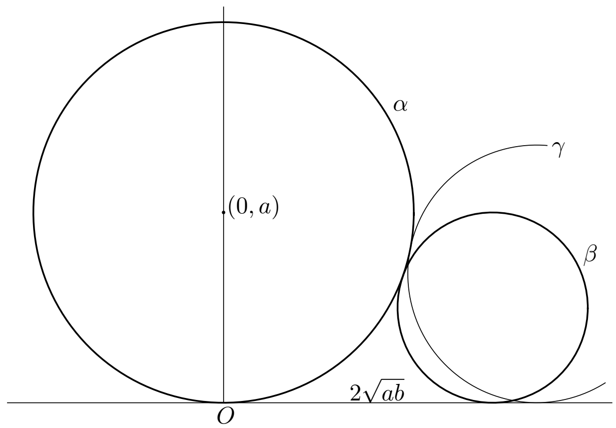

Figure 7.

4.1. The case $b \neq 0$ by division by zero calculus. We assume $b \neq 0$. Then we get[2]

(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x - 2\sqrt{ab})^2 + (y - b)^2 - b^2$,

(iii) $C_1 = -4a\left(\dfrac{x}{2\sqrt{ab}} + \dfrac{y}{2a} - 1\right)$,

(iv) $C_n = \dfrac{2\sqrt{ab}\left(-\frac{1}{b}\right)^n \left(\frac{1}{2}\right)_{n-1}}{\Gamma(n+1)} x$ for $n = 2, 3, 4, \cdots$, where $(x)_n$ is the Pochhammer symbol, i.e., $(x)_n = x(x+1)(x+2)\cdots(x+n-1)$.

Therefore the equation $C_0 = 0$ represents the circle β. The equation $C_1 = 0$ represents the line joining the farthest point on α from the x-axis and the point of tangency of β and the x-axis. Let s_0 be this line. The equation $C_n = 0$ represents the y-axis for $n = 2, 3, 4, \cdots$. The figures represented by $C_0 = 0$, $C_1 = 0$, $C_n = 0$ $(n = 2, 3, 4, \cdots)$ are denoted in Figure 8 in red.

For a circle δ of radius r and a line l whose distance from the center of δ equals d, we call d/r the cosine of the angle formed by δ and l and denote by $\cos(l, \delta)$:

$$\cos(l, \delta) = \frac{d}{r}. \tag{4.2}$$

If they intersect, it is actually the cosine of the angle between them.

The line s_0 passes through the point of tangency of α and β, because α and β are similar and the internal center of similitude coincides with the point of tangency of α and β, while the farthest point on α from the x-axis and the point of tangency of β and the x-axis are corresponding by the similarly. The circle β and the y-axis touch α and the x-axis by Proposition 1.1, but the line s_0 does not, but makes the same angle with them, where the cosine of the angle equals $\sqrt{b/(a+b)}$.

[2]We use Wolfram Mathematica to get C_n, but it does not behave properly sometimes.

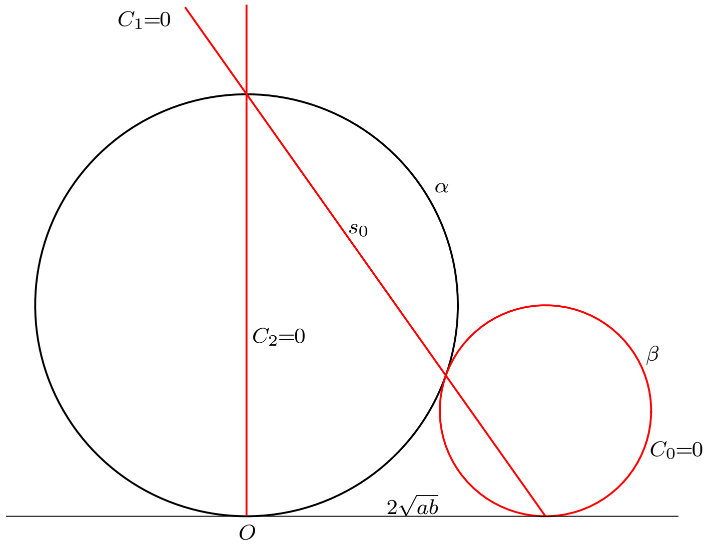

Figure 8.

4.2. The case $b = 0$ by division by zero. We consider the case $b = 0$ by division by zero. The equation (4.1) is arranged as follows in three ways;

$$(x^2 + y^2) - 4\sqrt{ar}x + 2r(2a - y) = 0,$$
$$\frac{x^2 + y^2}{\sqrt{r}} - 4\sqrt{a}x + 2\sqrt{r}(2a - y) = 0,$$
$$\frac{x^2 + y^2}{r} - 4\sqrt{\frac{a}{r}}x + 2(2a - y) = 0.$$

Therefore in the case $r = 0$, we have $x^2 + y^2 = 0$, $x = 0$, $y = 2a$ by (1.1), which represent the origin O, the y-axis and the tangent of α at the farthest point on α from the x-axis. The three figures are described in Figure 9 in red.

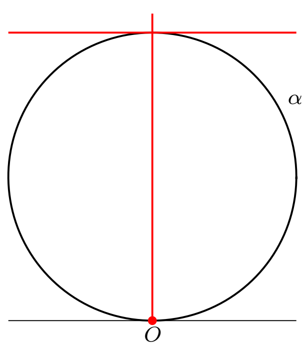

Figure 9: $b = 0$.

The line s_0 in the previous subsection corresponds to the tangent of α at the farthest point from the x-axis. For the line s_0 is represented by the equation $x/(2\sqrt{ab}) + y/(2a) = 1$, and by (1.1) it coincides with $y/(2a) = 1$ when $b = 0$. Or simply consider the line represented by $x/p + y/q = 1$ in the case $p = 0$.

5. A circle touching a circle and its secant

Let ε be a circle of diameter AU and center O, where $|AO| = a$, and let t be a secant of ε meeting in points T and U. We use a rectangular coordinate system with origin O such that A has coordinates $(a, 0)$, and T lies in the region $y > 0$. For a point Z of coordinates $(z, 0)$ on the line AU, let F be the foot of perpendicular from Z to t. We assume that δ_z is the circle touching t at F and the minor arc of ε cut by t if Z lies between A and U, otherwise δ_z is the circle touching ε externally and the line t at F from the side opposite to the minor arc of ε (see Figures 10 and 11). We consider the circle δ_a, i.e., we would like to consider the case in which the point F coincides with the point T. A similar situation is considered in [23].

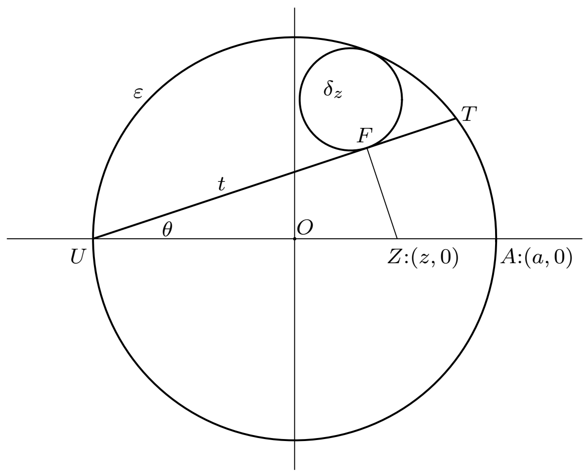

Figure 10.

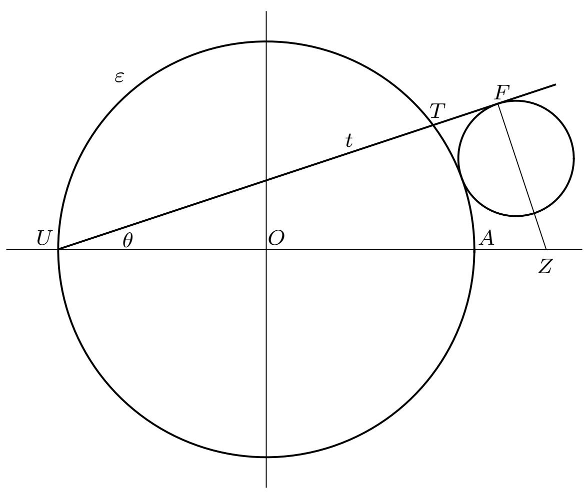

Figure 11.

Let θ be the angle between the line t and the x-axis and $m = \tan\theta$. Then t has an equation $t(x, y) = (x + a)m - y = 0$ and ZF has an equation $z_F(x, y) =$

$(x-z)+my=0$. Assume that δ_z has radius r and center of coordinates (p,q). Firstly assume Z lying between A and U. If q' is the y-coordinate of the point of intersection of t and the perpendicular from the center of δ_z to the x-axis, then there is a positive real number k such that $q=q'+k$. Then $t(p,q)=t(p,q')-k=-k<0$. Therefore we have

$$t(p,q)/\sqrt{1+m^2}=-r, \quad z_F(p,q)=0 \ \ \textit{and} \ \ p^2+q^2=(a-r)^2. \tag{5.1}$$

Let

$$v=\frac{a^2-z^2}{2a\sqrt{1+m^2}} \ \textit{and} \ w=\frac{(a+z)^2}{2a(1+m^2)}.$$

Solving (5.1) for p, q and r, we have

$$(p,q)=\left(w-mv-\frac{a^2+z^2}{2a}, \ v+mw\right), \ \textit{and} \ r=-mv+\frac{a^2-z^2}{2a}. \tag{5.2}$$

If Z does not lie between A and U, we have $t(p,q)/\sqrt{1+m^2}=r$, $z_F(p,q)=0$ and $p^2+q^2=(a+r)^2$, which are obtained from (5.1) by changing the signs of r. Therefore the solutions of these three equations are also obtained from (5.2) by changing the sign of r.

Therefore in any case, the circle δ_z is represented by the following equation using (5.2) with parameter z:

$$\delta_z(x,y)=(x-p)^2+(y-q)^2-r^2.$$

We now consider the Laurent expansion of $\delta_z(x,y)$ about $z=a$:

$$\delta_z(x,y)=\sum_{n=-\infty}^{\infty} C_n(z-a)^n.$$

Then we get
(i) $\cdots=C_{-3}=C_{-2}=C_{-1}=0$, (ii) $C_0=(x-a\cos 2\theta)^2+(y-a\sin 2\theta)^2$,
(iii) $C_1=2(-(\cos 2\theta+\sin\theta)x+(\cos\theta-\sin 2\theta)y+(1-\sin\theta)a)$,
(iv) $C_2=(x\sin\theta-y\cos\theta-a)(\sin\theta-1)/a$,
(v) $C_3=C_4=C_5=\cdots=0$.

Therefore the equation $C_0=0$ represents the point T. The equation $C_1=0$ represents the line TV, where V is the midpoint of the major arc of ε cut by t, whose coordinates are $(a\sin\theta,-a\cos\theta)$. The equation $C_2=0$ represents the tangent of the circle ε at the point V. The figures obtained by $C_n=0$ $(n=0,1,2)$ are described in Figure 12 in red.

The line TV forms the same angle with ε and t, which equals $\theta+\phi$, where $\phi=\angle TVO$. While $2\phi+\theta=\pi/2$, i.e., $\phi=(\pi/2-\theta)/2$. Therefore the same angle equals $\pi/4+\theta/2$. We can consider that the point T and the tangent of ε at V touch both ε and t, but the line TV does not. However it forms the same angle $\pi/4+\theta/2$ with ε and t.

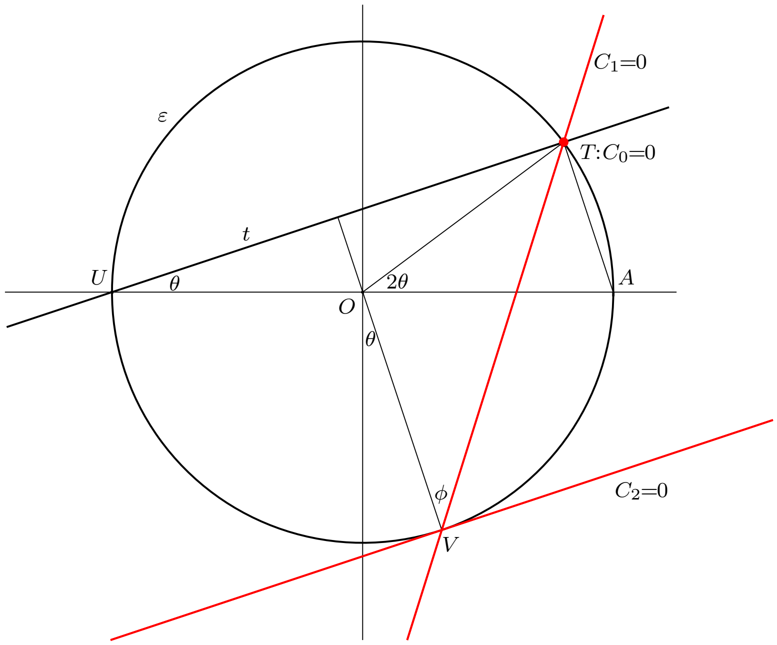

Figure 12.

6. Arbelos

For a point C on the segment AB such that $|BC| = 2a$, $|CA| = 2b$ and $|AB| = 2c$, let α, β and γ be circles of diameters BC, CA and AB, respectively. Each of the two congruent figures surrounded by the three circles is called an arbelos in a narrow sense and the radical axis of α and β is called the axis. Notice that $c = a + b$. We use a rectangular coordinate system with origin C such that A has coordinates $(-2b, 0)$ (see Figure 13).

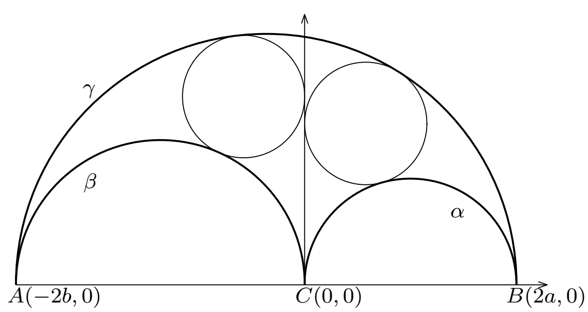

Figure 13.

If a circle touches one of given two circles internally and the other externally, we say that the circle touches the two circles in the opposite sense, otherwise in the same sense. The two circles touching α and γ in the opposite sense and the axis from the side opposite to A are congruent to the two circles touching β and γ in the opposite sense and the axis from the side opposite to B, and have common radius

$r_{\mathrm{A}} = ab/c$. It is believed that the congruent circles were studied by Archimedes and circles of radius r_{A} are said to be Archimedean.

6.1. The twin circles of Archimedes. Usually the arbelos is described by three semicircle as the upper half part of the figure as in Figure 13. In this case the two Archimedean circles touching γ and one of α and β in the opposite sense are called the twin circles of Archimedes. The circle of center of coordinates $(ma, 0)$ (resp. $(-nb, 0)$) and passing through the point C is denoted by α_m (resp. β_n) for real numbers m and n. A circle touching γ internally and touching α_m and β_n in the region $y \geq 0$ is said to touch the three circles appropriately if the points of tangency of this circle and each of α_m, γ and β_n lie counterclockwise (see Figure 14). We consider the next theorem.

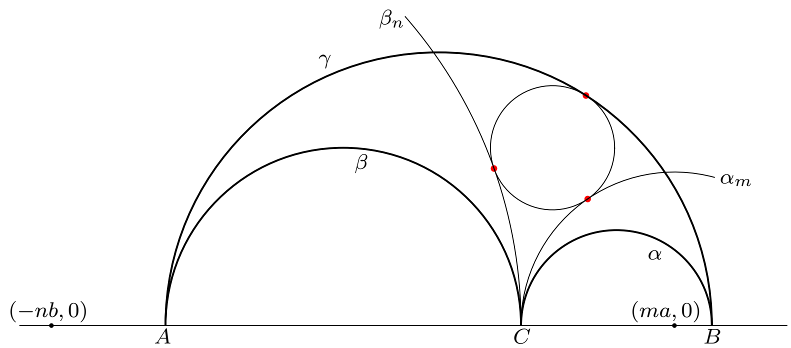

Figure 14.

Theorem 6.1 ([27])**.** *Assume $(m, n) \neq (0, 1), (1, 0)$. A circle touching α_m, β_n and γ appropriately is Archimedean if and only if*

$$\frac{1}{m} + \frac{1}{n} = 1. \tag{6.1}$$

The theorem characterizes the Archimedean circles touching γ internally, but the twin circles of Archimedes are excluded. In this subsection, we show that the twin circles can be included in the theorem by division by zero. We consider the case $(m, n) = (1, 0)$. The circle β_n has an equation $(x + nb)^2 + y^2 = (nb)^2$ or

$$x^2 + y^2 + 2nbx = 0. \tag{6.2}$$

Therefore we get $x^2 + y^2 = 0$ if $n = 0$, i.e., β_0 coincides with the origin. On the other hand (6.2) implies

$$\frac{x^2 + y^2}{n} + 2bx = 0.$$

Therefore we get $x = 0$ if $n = 0$ by (1.1). Therefore $n = 0$ implies that β_0 is the origin or the y-axis. Since the origin is a part of the y-axis, we can consider that β_0 is the y-axis. While α_1 coincides with the circle α. Hence $(m, n) = (1, 0)$ satisfies (6.1) and we get one of the twin circles of Archimedes touching α. Similarly in the case $(m, n) = (0, 1)$ we get the other Archimedean circle touching $\beta_1 = \beta$ and the axis.

6.2. Parametric representation. Let $d = \sqrt{ab}/c$. We use the next theorem.

Theorem 6.2. *The following statements hold.*
(i) *A circle touches the circles α and β in the same sense if and only if its has radius r_z^γ and center of coordinates (x_z^γ, y_z^γ) given by*

$$q_z^\gamma = \frac{abc}{c^2z^2 - ab}, \quad r_z^\gamma = |q_z^\gamma| \quad and \quad (x_z^\gamma, y_z^\gamma) = \left(\frac{b-a}{c} q_z^\gamma, 2zq_z^\gamma\right)$$

for a real number $z \neq \pm d$.
(ii) *A circle touches the circles β and γ in the opposite sense if and only if it has radius r_z^α and center of coordinates (x_z^α, y_z^α) given by*

$$r_z^\alpha = \frac{abc}{a^2z^2 + bc} \quad and \quad (x_z^\alpha, y_z^\alpha) = \left(-2b + \frac{b+c}{a} r_z^\alpha, 2zr_z^\alpha\right)$$

for a real number z.
(iii) *A circle touches the circles γ and α in the opposite sense if and only if it has radius r_z^β and center of coordinates (x_z^β, y_z^β) given by*

$$r_z^\beta = \frac{abc}{b^2z^2 + ca} \quad and \quad (x_z^\beta, y_z^\beta) = \left(2a - \frac{c+a}{b} r_z^\beta, 2zr_z^\beta\right)$$

for a real number z.

Proof. Let γ_z be the circle of radius and center described in (i). Then we have $(x_z^\gamma - a)^2 + (y_z^\gamma)^2 = (a + q_z^\gamma)^2$. Therefore γ_z and α touch internally or externally according as $q_z^\gamma < 0$ or $q_z^\gamma > 0$. Similarly γ_z and β touch internally or externally according as $q_z^\gamma < 0$ or $q_z^\gamma > 0$. Hence γ_z touches α and β in the same sense. Conversely we assume that a circle γ' of radius $r > 0$ touches α and β in the same sense. Then there is a real numbers z such that $r_{\pm z}^\gamma = r$. Therefore we have $\gamma' = \gamma_z$ or $\gamma' = \gamma_{-z}$. This proves (i). The rest of the theorem can be proved similarly. □

Essentially the same formulas as Theorem 6.2 can be found in [30], not so simple though. Simpler expression in the case z being an integer can be found in [4, 5]. We denote the circle of radius r_z^α and center of coordinates (x_z^α, y_z^α) by α_z. Also the equation representing the circle α_z is denoted by $\alpha_z(x, y) = 0$, where

$$\alpha_z(x, y) = (x - x_z^\alpha)^2 + (y - y_z^\alpha)^2 - (r_z^\alpha)^2. \tag{6.3}$$

The circles β_z and γ_z and the equations $\beta_z(x, y) = 0$ and $\gamma_z(x, y) = 0$ are defined similarly, respectively. The circles α_0, β_0 and γ_0 coincide with the circles α, β and γ, respectively. While $\alpha_1 = \beta_1 = \gamma_1$ (resp. $\alpha_{-1} = \beta_{-1} = \gamma_{-1}$) is the incircle of the arbelos in the region $y \geq 0$ (resp. $y \leq 0$). The Archimedean circles touching α and γ in the opposite sense coincide with the circles $\beta_{\pm\sqrt{\frac{c}{b}}}$. Also the Archimedean circles touching β and γ in the opposite sense coincide with the circles $\alpha_{\pm\sqrt{\frac{c}{a}}}$.

The circle γ_z touches α and β internally (resp. externally) if and only if $|z| < d$ (resp. $|z| > d$), which is also equivalent to $q_z^\gamma < 0$ (resp. $q_z^\gamma > 0$). The external common tangents of α and β are also denoted by $\gamma_{\pm d}$ and have the following equations [25, 26]:

$$\gamma_{\pm d}(x, y) = (a - b)x \mp 2\sqrt{ab}y + 2ab = 0. \tag{6.4}$$

Theorem 6.3. *The following statements hold.*
(i) *Circles or a circle and a line γ_z and γ_w touch if and only if $|z-w|=1$.*
(ii) *Circles α_z and α_w touch if and only if $|z-w|=1$.*
(iii) *Circles β_z and β_w touch if and only if $|z-w|=1$.*

Proof. If γ_z and γ_w are circles, then the part (i) follows from

$$(x_z^\gamma - x_w^\gamma)^2 + (y_z^\gamma - y_w^\gamma)^2 - (q_z^\gamma + q_w^\gamma)^2 = \frac{4a^2b^2c^2((z-w)^2-1)}{(c^2z^2-ab)(c^2w^2-ab)}.$$

We consider the case in which γ_z touches γ_d. In this case γ_z touches γ_d from the same side as the point C and touches α and β externally and $q_z^\gamma > 0$ (see Figure 15). Let p be the y-coordinate of the point of intersection of γ_d and the perpendicular from the center of γ_z to the x-axis. Then there is a real number $k>0$ such that $p = y_z^\gamma + k$. Hence we have

$$\gamma_d(x_z^\gamma, y_z^\gamma) = \gamma_d(x_z^\gamma, p-k) = \gamma_d(x_z^\gamma, p) + 2\sqrt{ab}k = 2\sqrt{ab}k > 0$$

by (6.4). Therefore γ_z touches γ_d if and only if $\gamma_d(x_z^\gamma, y_z^\gamma)/c = q_z^\gamma$. While we have

$$\frac{\gamma_d(x_z^\gamma, y_z^\gamma)}{c} - q_z^\gamma = \frac{2abc(z-(d+1))(z-(d-1))}{(c^2z^2-ab)}.$$

Therefore γ_z touches γ_d if and only if $z = d \pm 1$. The case γ_z touching γ_{-d} is proved similarly. This proves (i). The circles α_z and α_w touch if and only if they touch externally. Therefore the part (ii) follows from

$$(x_z^\alpha - x_w^\alpha)^2 + (y_z^\alpha - y_w^\alpha)^2 - (r_z^\alpha + r_w^\alpha)^2 = \frac{4a^2b^2c^2((z-w)^2-1)}{(a^2z^2+bc)(a^2w^2+bc)}.$$

The part (iii) is proved similarly. □

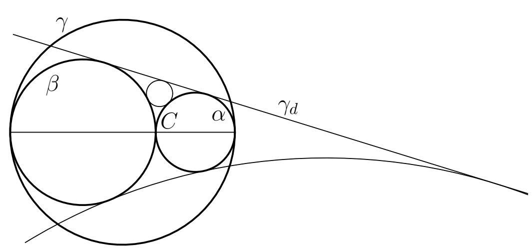

Figure 15.

6.3. Center of similitude of two circles. In this subsection we demonstrate fundamental results on the centers of similitude of two of the three circles α, β and γ, which have been getting little concern on the study of the arbelos. The results were discovered from the consideration of circles touching the two circles using the parametric representation stated in the previous subsection by division by zero calculus. We omit the proofs since they are straightforward.

Let S_a (resp. S_b) be the internal center of similitude of the circles β (resp. α) and γ, and let S_c be the external center of similitude of the circles α and β. The points have coordinates

$$S_a : \left(\frac{-2b^2}{b+c}, 0\right), \ S_b : \left(\frac{2a^2}{c+a}, 0\right), \ S_c : \left(\frac{2ab}{b-a}, 0\right). \tag{6.5}$$

The perpendiculars to AB at the three points are denoted by s_a, s_b and s_c, respectively. If $a = b$, S_c and s_c coincide with the origin and the axis, respectively by (1.1). Recall (4.2).

Theorem 6.4. *The following relations hold.*
(i) $\cos(s_a, \beta) = \cos(s_a, \gamma) = \dfrac{a}{b+c}$. (ii) $\cos(s_b, \gamma) = \cos(s_b, \alpha) = \dfrac{b}{c+a}$.
(iii) $\cos(s_c, \alpha) = \cos(s_c, \beta) = \dfrac{c}{|b-a|}$ *if* $a \neq b$.

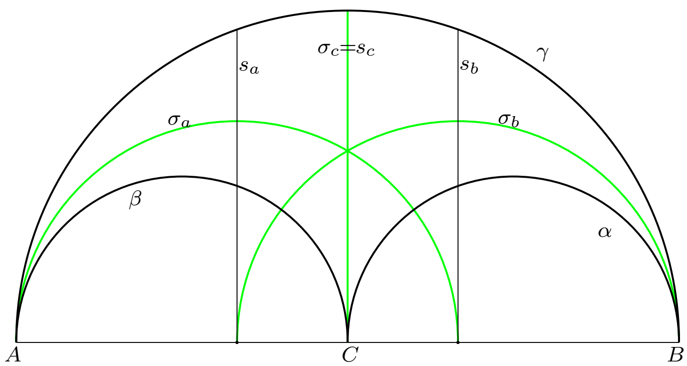

Figure 16: $a = b$.

Let σ_a be the circle of center S_a passing through the point A. Similarly the circles σ_b and σ_c are defined. The circle σ_c is represented by the equation $4abx/(a-b) + x^2 + y^2 = 0$ or $4abx + (a-b)(x^2 + y^2) = 0$. This implies that $x^2 + y^2 = 0$ or $x = 0$ if $a = b$, i.e., σ_c *coincides with the* y*-axis if* $a = b$. Some results on the point S_c and the circle σ_c can be found in [18, 19]. For two circles δ_1 and δ_2 of radii r_1 and r_2, we define the cosine of the angle made by the two circles by

$$\cos(\delta_1, \delta_2) = \frac{r_1^2 + r_2^2 - d^2}{2r_1 r_2}, \tag{6.6}$$

where d is the distance between the centers of the two circles.

Theorem 6.5. *The following statements hold.*
(i) *The circle* σ_a *(resp.* σ_b, σ_c*) is orthogonal to the circle* α_z *(resp.* β_z, γ_z*) for any real number* z.
(ii) $\cos(\sigma_a, \sigma_b) = |\cos(\sigma_b, \sigma_c)| = |\cos(\sigma_c, \sigma_a)| = \dfrac{1}{2}$.

Notice that Theorem 6.5(ii) holds in the case $a = b$ (see Figure 16). By Theorem 6.5(i), the circles α_z and α_w $(z \neq w)$ are fixed by the inversion in the circle σ_a. Therefore their radical axis is also fixed, i.e., it passes through the point S_a. We get the next theorem, where recall Theorem 6.3 (see Figure 17).

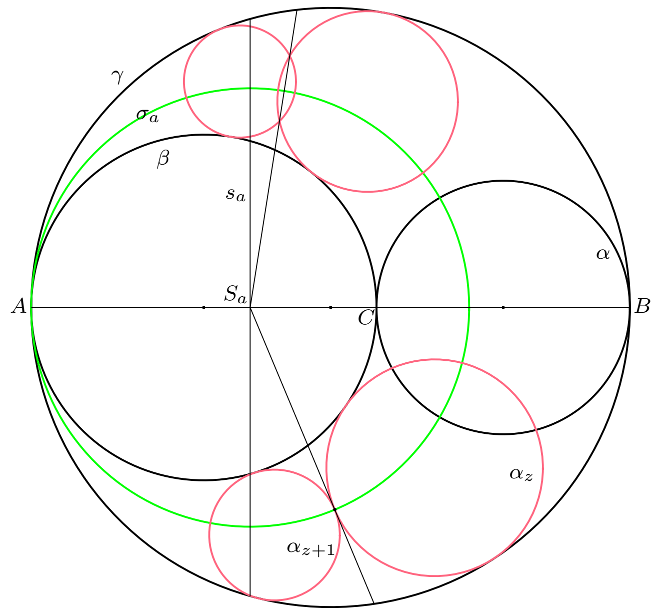

Figure 17.

Theorem 6.6. *The radical axis of α_z and α_w passes through the point S_a for real numbers z and w ($z \neq w$). In particular, the point of tangency of α_z and α_{z+1} lies on the circle σ_a and the common tangent at the point passes through S_a. Similar statements hold for S_b and σ_b and also for S_c and σ_c.*

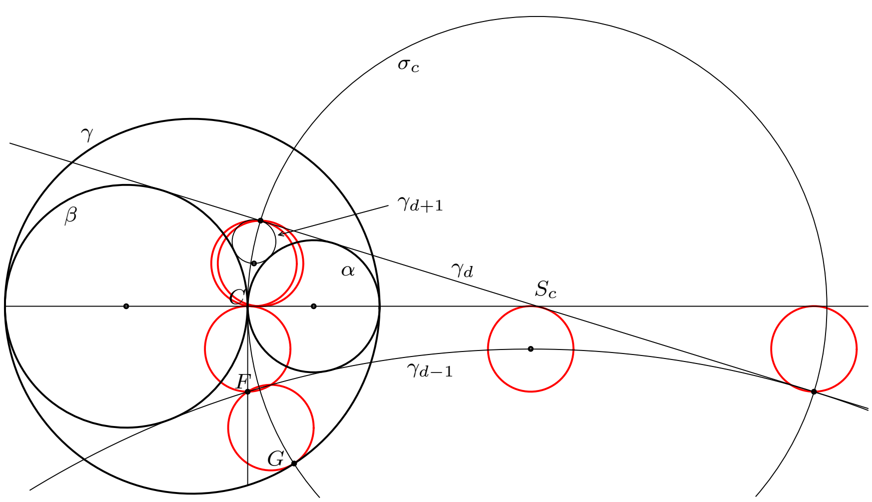

Figure 18: Archimedean circles related to the circles $\gamma_{d\pm 1}$.

Theorem 6.7. *The circles σ_a, σ_b and σ_c belong to the same pencil of circles and pass through the points of coordinates given by*

$$\Sigma^+ : \left(\frac{ab(b-a)}{c^2-ab}, \frac{\sqrt{3}abc}{c^2-ab}\right), \quad \Sigma^- : \left(\frac{ab(b-a)}{c^2-ab}, -\frac{\sqrt{3}abc}{c^2-ab}\right). \tag{6.7}$$

In this paragraph we consider some new Archimedean circles without division by zero and division by zero calculus. Assume $a \neq b$. The circle γ_{d-1} meets the axis in two points, and the closer point to C is denoted by F. Assume σ_c meets γ in a point G in the region $y < 0$. Recall $r_{\mathrm{A}} = ab/c$. The following statements hold, where some of the statements can be found in [9] (see Figure 18):
(i) The distance from the point of tangency of γ_d and $\gamma_{d\pm1}$ to AB equals $2r_{\mathrm{A}}$.
(ii) $|CF| = 2r_{\mathrm{A}}$,
(iii) $|FG| = 2r_{\mathrm{A}}$ and the Archimedean circle of diameter FG touches γ at G.
(iv) There are two Archimedean circles whose center coincide with one of the closest points on $\gamma_{d\pm1}$ to AB such that they touch the lines γ_d and AB.

7. Circles touching two given circles forming the arbelos

In this section we consider the circles α_z and γ_z by division be zero calculus. The highlight of this section is that the line s_a (resp. s_c) is derived by considering α_z (resp. γ_z) by division by zero calculus.

7.1. The circle α_z. We consider the circle α_z represented by the equation (6.3). If we consider $\alpha_z(x, y)$ as a function of z, there is no singular case. We firstly consider the Laurent expansion of $\alpha_z(x, y)$ about $z = 0$:

$$\alpha_z(x, y) = \sum_{n=-\infty}^{\infty} C_n z^n.$$

Then we get
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x-a)^2 + y^2 - a^2$,
(iii) $C_{2n-1} = (-1)^n \dfrac{4a^{2n-1}}{(bc)^{n-1}} y$ for $n = 1, 2, 3, \cdots$,
(iv) $C_{2n} = (-1)^{n-1} \dfrac{2a^{2n}(b+c)}{(bc)^n} \left(x + \dfrac{2b^2}{b+c}\right)$ for $n = 1, 2, 3, \cdots$.

We consider the figures represented by the equation $C_n = 0$. Then $C_0 = 0$ implies the equation $(x-a)^2 + y^2 = a^2$, which represents the circle α. The equations $C_1 = C_3 = C_5 = \cdots = 0$ imply $y = 0$, which represents the line AB. And the equations $C_2 = C_4 = C_6 = \cdots = 0$ represent the line s_a by (6.5). The three figures represented by $C_n = 0$ are described in Figure 19 in red. Notice that the circle α can be obtained in the usual way from the equation (6.3), but the lines s_a and AB can not.

The line AB is orthogonal to β and γ. Hence we can consider it touches the two circles by Proposition 1.1, i.e., AB is eligible to be a figure touching β and γ. However the line s_a does not touch the two circles, but intersects at the same angle by Theorem 6.4.

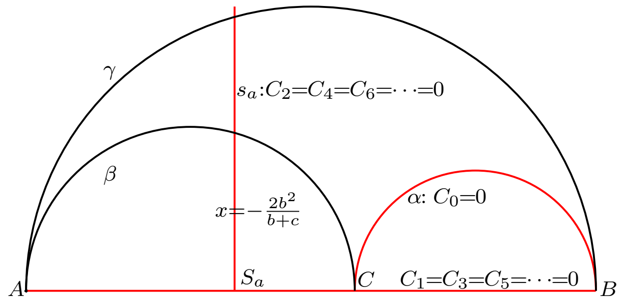

Figure 19.

Let $w = 1/z$ and $\alpha_w(x, y) = \alpha_z(x, y)$. We consider the Laurent expansion of α_w about $w = 0$:

$$\alpha_w(x, y) = \sum_{n=-\infty}^{\infty} C_n w^n.$$

Then we have:

(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x + 2b)^2 + y^2$,

(iii) $C_{2n-1} = (-1)^n \dfrac{4(bc)^n}{a^{2n-1}} y$ for $n = 1, 2, 3, \cdots$,

(iv) $C_{2n} = (-1)^n \dfrac{2(bc)^n(b+c)}{a^{2n}} \left(x + \dfrac{2b^2}{b+c} \right)$ for $n = 1, 2, 3, \cdots$.

Therefore the equation $C_0 = 0$ represents the point A instead of the circle α, and $C_{2n-1} = 0$ represents the line AB for $n = 1, 2, 3, \cdots$, and $C_{2n} = 0$ represents the line s_a for $n = 1, 2, 3, \cdots$. The three figures obtained from $C_n = 0$ are described in Figure 20 in red.

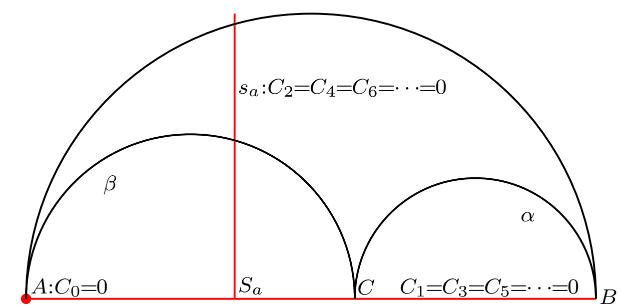

Figure 20.

7.2. The circle γ_z in the case $z = 0$. The circle γ_z has an equation

$$\gamma_z(x, y) = (x - x_z^\gamma)^2 + (y - y_z^\gamma)^2 - (r_z^\gamma)^2 = 0$$

for a real number $z \neq \pm d$ by Theorem 6.2. We consider the Laurent expansion of $\gamma_z(x, y)$ about $z = 0$:

$$\gamma_z(x, y) = \sum_{n=-\infty}^{\infty} C_n z^n.$$

Then we get

(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x-2a)(x+2b)+y^2$,

(iii) $C_{2n-1} = \dfrac{4c^{2n-1}}{(ab)^{n-1}} y$ for $n = 1, 2, 3, \cdots$,

(iv) For $n = 1, 2, 3, \cdots$, we have

$$\begin{aligned} C_{2n} &= -\frac{2c^{2n}(a-b)}{(ab)^n}\left(x - \frac{2ab}{b-a}\right) \quad if \quad a \neq b, \\ C_{2n} &= -4^{n+1}a^2 \quad if \quad a = b. \end{aligned}$$

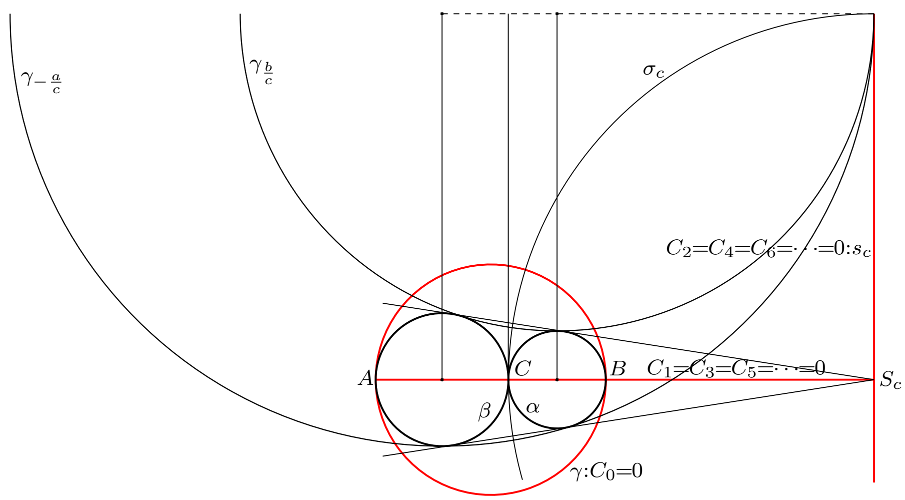

Figure 21: $a < b$.

The equation $C_0 = 0$ represents the circle γ. The equations $C_1 = C_3 = C_5 = \cdots = 0$ represent the line AB. The equations $C_2 = C_4 = C_6 = \cdots = 0$ represent the line s_c if $a \neq b$. If $a = b$, they represent no figure. Notice that s_c does not touch α and β but we have $\cos(s_c, \alpha) = \cos(s_c, \beta)$ by Theorem 6.4.

Assume $a \neq b$. The three figures obtained from $C_n = 0$ in this case are described in Figure 21 in red. The circles $\gamma_{\mp a/c}$ and $\gamma_{\pm b/c}$ touch by Theorem 6.3. The points of tangency coincide with the points of intersection of s_c and σ_c. Therefore each of the points of tangency, S_c and C form three vertices of a square. The three center of the circles $\gamma_{\pm b/c}$ and α lie on a perpendicular to AB by Theorem 6.2. Also the centers of the circles $\gamma_{\pm a/c}$ and β lie on a perpendicular to AB. If $a = b$, then $d = 1/2$ and $\gamma_{\pm b/c} = \gamma_{\pm a/c} = \gamma_{\pm 1/2}$ are the external common tangents of α and β parallel to AB.

7.3. $w = 1/z$ **and** $\gamma_w(x,y) = \gamma_z(x,y)$ **in the case** $w = 0$**.** Let $w = 1/z$ and $\gamma_w(x,y) = \gamma_z(x,y)$. We consider the case $w = 0$ using the Laurent expansion of $\gamma_w(x,y)$ about $w = 0$:

$$\gamma_w(x,y) = \sum_{n=-\infty}^{\infty} C_n w^n.$$

Then we get
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = x^2 + y^2$,
(iii) $C_{2n-1} = -\dfrac{4a^nb^n}{c^{2n-1}}y$ for $n = 1, 2, 3\cdots$,
(iv) For $n = 1, 2, 3, \cdots$, we have

$$\begin{aligned} C_{2n} &= \frac{2(ab)^n(a-b)}{c^{2n}}\left(x - \frac{2ab}{b-a}\right) \ if \ a \neq b, \\ C_{2n} &= 4^{1-n}a^2 \ if \ a = b. \end{aligned}$$

Therefore $C_0 = 0$ does not represent the circle γ but the origin. The others are the same as those in subsection 7.2. The figures represented by $C_n = 0$ in the case $a \neq b$ are denoted in Figure 22 in red .

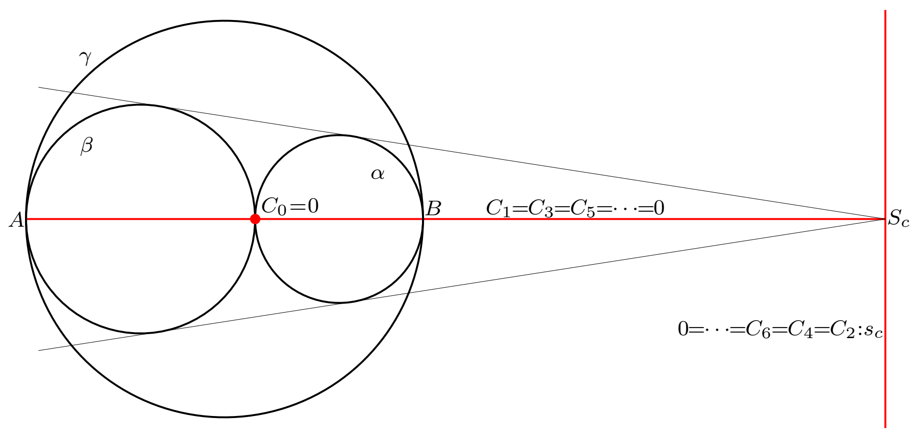

Figure 22.

7.4. The circles $\gamma_{b/c}$ and $\gamma_{-a/c}$ in the case $a = b$. We have seen that the circles $\gamma_{b/c}$ and $\gamma_{-a/c}$ touch the line s_c if $a \neq b$ in subsection 7.2. We now consider the relation between $\gamma_{b/c}$, $\gamma_{-a/c}$ and s_c in the case $a = b$. If $a = b$, then $\gamma_{b/c} = \gamma_{1/2} = \gamma_d$ and $\gamma_{-a/c} = \gamma_{-1/2} = \gamma_{-d}$. In this subsection we firstly assume $a = b$ and consider the circles $\gamma_{1/2}$ and $\gamma_{-1/2}$. Secondly we drop the assumption $a = b$ and consider $\gamma_{b/c}(x, y)$ and $\gamma_{-a/c}$ in the case $a = b$.

It seems that the figures obtained by the Laurent expansion of $\gamma_z(x, y)$ about $z = \pm 1/2$ in the case $a = b$ can be obtained if we consider the Laurent expansion of $\gamma_z(x, y)$ about $z = \pm d$ in the case $a \neq b$ then consider the resulting figure in the case $a = b$. We will see this result in subsection 7.5.

Assume $a = b$. We consider the Laurent expansion of $\gamma_z(x, y)$ about $z = 1/2$:

$$\gamma_z(x, y) = \sum_{n=-\infty}^{\infty} C_n \left(z - \frac{1}{2}\right)^n .$$

Then we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$, (ii) $C_{-1} = a(a-y)$,
(iii) $C_0 = x^2 + \left(y - \dfrac{a}{2}\right)^2 - \left(\dfrac{\sqrt{5}a}{2}\right)^2$, (iv) $C_n = (-1)^{n+1}(a+y)$ for $n = 1, 2, \cdots$.

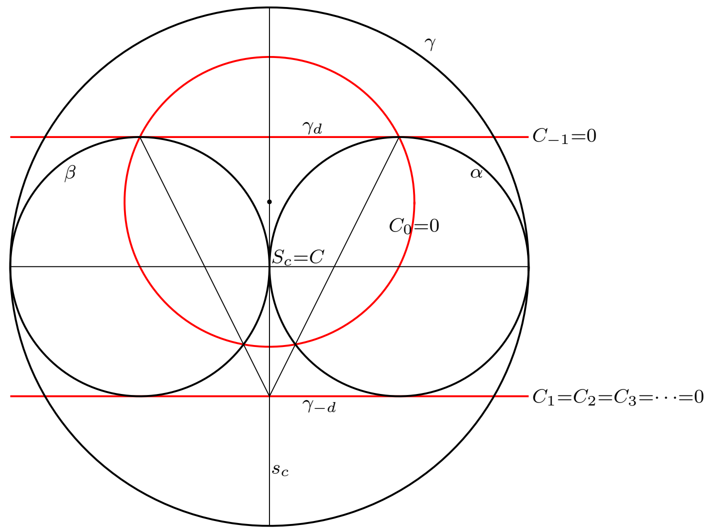

Figure 23.

Hence $C_{-1} = 0$ represents the line $\gamma_d = \gamma_{1/2}$. The equation $C_0 = 0$ represents the circle of radius $\sqrt{5}a/2$ and center of coordinates $(0, a/2)$. The circle passes through the point of tangency of γ_d and each of α and β. The radical center of this circle and α and β coincides with the point of intersection of the axis and γ_{-d}. Since the point S_c coincides with the origin C and s_c coincides with the axis, the circle touches the line s_c by Proposition 1.1. The equations $C_1 = C_2 = C_3 = \cdots = 0$ represent γ_{-d}. The three figures obtained by $C_n = 0$ touch the line s_c and are described in Figure 23 in red. Considering the Laurent expansion of $\gamma_z(x, y)$ about $z = -1/2$, we get the figures which are the reflection of the three figures in the line AB.

We now drop the assumption $a = b$ and consider $\gamma_{b/c}(x, y)$ as a function of b and its Laurent expansion about $b = a$:

$$\gamma_{b/c}(x, y) = \sum_{n=-\infty}^{\infty} C_n (b-a)^n.$$

Then we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$, (ii) $C_{-1} = 4a^2(a-y)$,
(iii) $C_0 = (x-a)^2 + (y-2a)^2 - a^2$, (iv) $C_1 = C_2 = C_3 = \cdots = 0$.
Therefore $C_{-1} = 0$ represents γ_d. The equation $C_0 = 0$ represents the circle of radius a and center of coordinates $(a, 2a)$. The two figures obtained by $C_{-1} = 0$ and $C_0 = 0$ touch the line s_c and are described in Figure 24 in red.

We consider $\gamma_{-a/c}(x,y)$ as a function of b and its Laurent expansion about $b=a$:

$$\gamma_{-a/c}(x,y)=\sum_{n=-\infty}^{\infty} C_n(b-a)^n.$$

Then we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$, (ii) $C_{-1} = -4a^2(a+y)$,
(iii) $C_0 = (x+a)^2 + (y-2a)^2 - (\sqrt{13}a)^2$, (iv) $C_1 = 2(x-2a)$.
(v) $C_2 = C_3 = C_4 = \cdots = 0$.
Therefore $C_{-1}=0$ represents γ_{-d}. The equation $C_0=0$ represents the circle of radius $\sqrt{13}a$ and the center of coordinates $(-a,2a)$. The circle passes through the point B and the point of tangency of α and γ_{-d}. The equation $C_1=0$ represents the tangent of α and γ at B. The three figures obtained by $C_{-1}=0$, $C_0=0$ and $C_1=0$ are described in Figure 24 in green. The circle represented by $C_0=0$ does not touch the line s_c, while the other two lines touch. The circle intersects γ, γ_{-d}, α and the tangents of γ at B at the same angle, whose cosine equals $3/\sqrt{13}$, where recall (4.2) and (6.6).

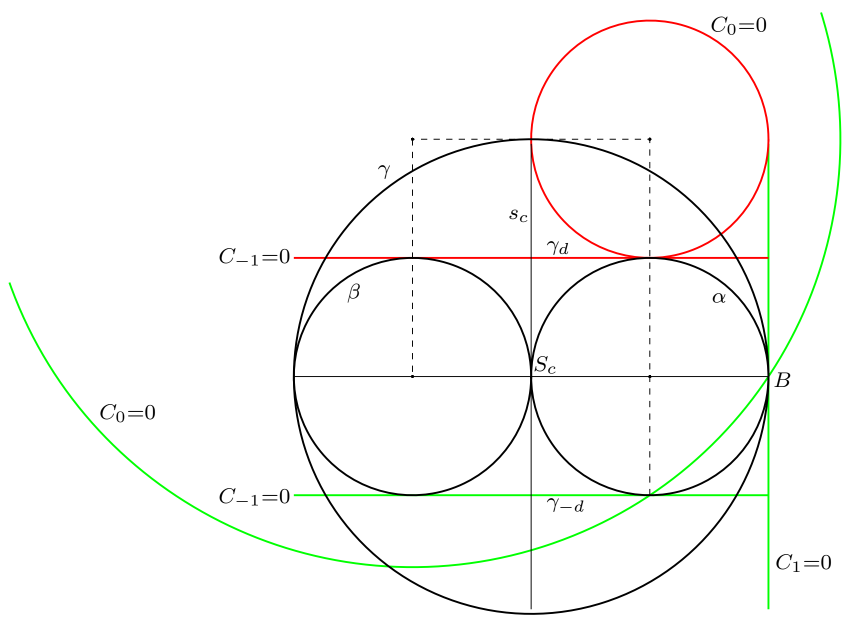

Figure 24: $\gamma_{b/c}$ (red) and $\gamma_{-a/c}$ (green) in the case $b=a$.

7.5. The circle γ_z in the case $z=d$. We consider the circle γ_z in the singular case $z=d=\sqrt{ab}/c$ and consider the Laurent expansion of $\gamma_z(x,y)$ about $z=d$:

$$\gamma_z(x,y)=\sum_{n=-\infty}^{\infty} C_n(z-d)^n.$$

Then we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$, (ii) $C_{-1} = d((a-b)x - 2\sqrt{ab}y + 2ab)$,

(iii) $C_0 = \left(x - \dfrac{a-b}{4}\right)^2 + \left(y - \dfrac{\sqrt{ab}}{2}\right)^2 - \left(\dfrac{\sqrt{a^2+18ab+b^2}}{4}\right)^2$,

(iv) $C_n = -\dfrac{1}{2}\left(\dfrac{-1}{2d}\right)^n ((a-b)x + 2\sqrt{ab}y + 2ab)$ for $n = 1, 2, 3, \cdots$.

Therefore the equation $C_{-1} = 0$ represents the line γ_d given by (6.4). The equation $C_0 = 0$ represents a circle. The radius and the coordinates of its center are given by

$$\frac{\sqrt{a^2+18ab+b^2}}{4}, \quad \left(\frac{a-b}{4}, \frac{\sqrt{ab}}{2}\right). \tag{7.1}$$

We denote this circle by $\overline{\gamma}$ and consider in detail in the next subsection. The equations $C_1 = C_2 = C_3 = \cdots = 0$ represent the line γ_{-d}. The figures obtained by $C_n = 0$ are described in Figure 25 in red. It is obvious that the circle $\overline{\gamma}$ coincides with the circle obtained in 7.4 if $a = b$ (see Figure 23).

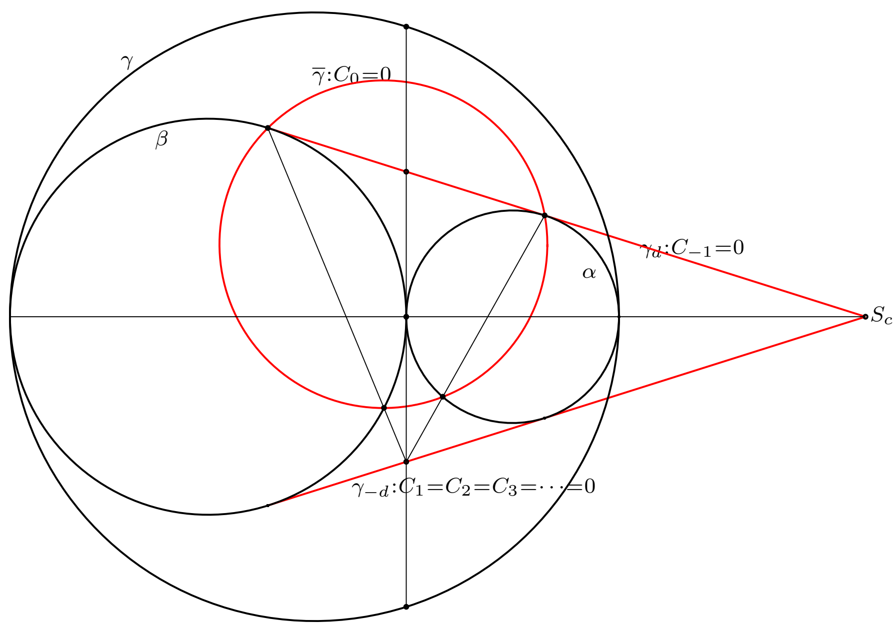

Figure 25.

7.6. The circle $\overline{\gamma}$. We consider the circle $\overline{\gamma}$ given in (7.1) in detail. Let I_i be the point of coordinates $(0, i\sqrt{ab})$ for an integer i. We use the next theorem (see Figure 25).

Theorem 7.1 ([25]). *The following statements are true.*
(i) *The point of tangency of γ_d and each of α and β lies on $\overline{\gamma}$.*
(ii) *The radical center of the three circles α, β and $\overline{\gamma}$ coincides with the point I_{-1}.*

The y-axis meets the circle γ and the lines $\gamma_{\pm d}$ in the points $I_{\pm 2}$ and $I_{\pm 1}$, respectively. While the radical axis of γ and $\overline{\gamma}$ passes through the point I_3. Hence we have the next theorem (see Figure 26).

Theorem 7.2. *The six points, where the y-axis meets γ, $\gamma_{\pm d}$, the line AB, the radical axis of γ and $\overline{\gamma}$, are evenly spaced.*

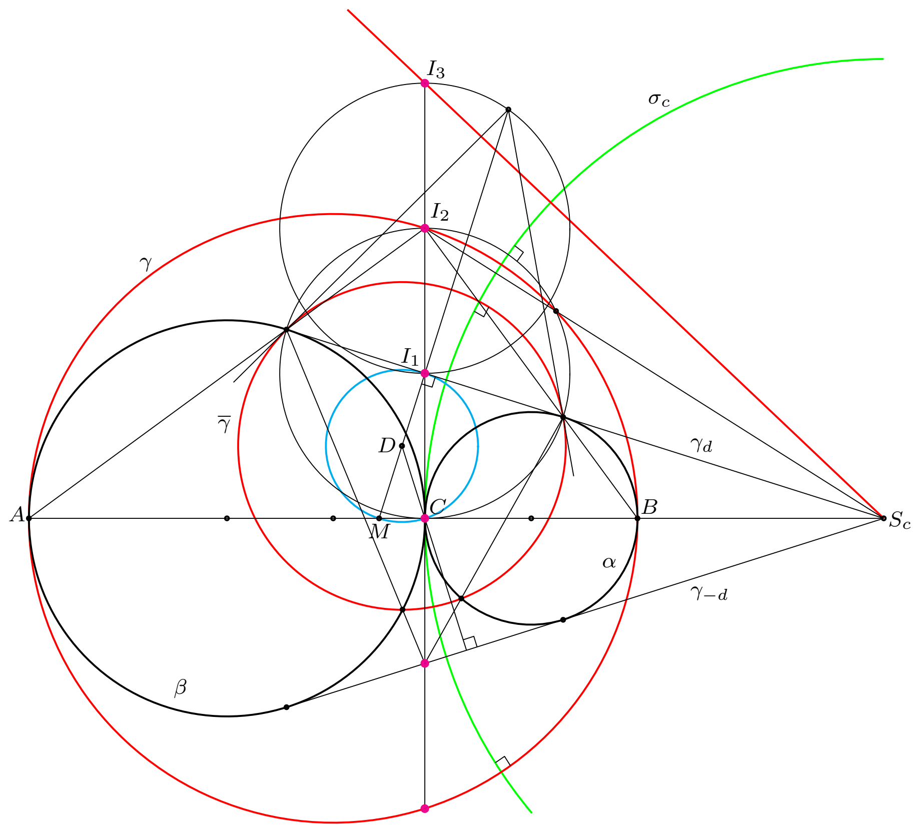

Figure 26.

The six points are described in Figure 26 in magenta. Let D be the center of the circle $\overline{\gamma}$, and let M be the midpoint of the segment joining C and the center of γ. It is well-known that the circle of diameter CI_2 passes through the point of tangency of γ_d and each of α and β and each of the two points lie on the segments AI_2 and BI_2. We have the next theorem, where recall (6.6) for the part (vii). The proofs are straightforward and are omitted.

Theorem 7.3. *The following statements are true.*
(i) *The circle σ_c is orthogonal to $\overline{\gamma}$ and the circle of diameter CI_2. Hence the radical axis of $\overline{\gamma}$ and γ_z passes through S_c for any real number z.*
(ii) *The circle of radius $c/4$ and center D passes though the points C, I_1 and M, and touches the line γ_d at I_1.*
(iii) *The lines CD and γ_{-d} are perpendicular.*
(iv) *The points D, I_1 and M are collinear.*
(v) *The line S_cI_2, the circle of diameter CI_2 and the circle γ meet in a point.*
(vi) *The line DI_1, the tangent of $\overline{\gamma}$ at the point of tangency of γ_d and each of α and β, and the circle of diameter I_1I_3 meet in a point.*

(vii) *The circle* $\overline{\gamma}$ *intersects* α *and* β *at the same angle, whose cosine equals* $\dfrac{c}{\sqrt{a^2+18ab+b^2}}$.

The circle $\overline{\gamma}$ is an iconic figure which shows how essential and interest things division by zero calculus brings us, and used in both the front and the back covers of the book [29].

7.7. Another parametric representation. We consider the circles touching α and β in the same sense using another parametric equation:

Theorem 7.4 ([26]). *A circle touching* α *and* β *in the same sense if and only if it is represented by the equation*

$$\zeta_z(x,y) = \left(x - \frac{b-a}{z^2-1}\right)^2 + \left(y - \frac{2z\sqrt{ab}}{z^2-1}\right)^2 - \left(\frac{c}{z^2-1}\right)^2 = 0 \qquad (7.2)$$

for a real number $z \neq \pm 1$.

We denote the circle by ζ_z.

7.7.1. *Case 1.* We firstly consider the circle ζ_z in the case $z=0$ using the Laurent expansion of $\zeta_z(x,y)$ about $z=0$:

$$\zeta_z(x,y) = \sum_{n=-\infty}^{\infty} C_n z^n.$$

Then we get
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x-2a)(x+2b)+y^2$,
(iii) $C_{2n-1} = 4\sqrt{ab}y$ for $n = 1,2,3,\cdots$,
(iv) $C_{2n} = 2\left((b-a)x - 2ab\right)$ for $n = 1,2,3,\cdots$.

Therefore the figures represented by $C_n = 0$ coincide with the figures represented by $C_n = 0$ in 7.2.

7.7.2. *Case 2.* Let $w = 1/z$ and $\zeta_w(x,y) = \zeta_z(x,y)$. We consider the case $w=0$ with the Laurent expansion of ζ_w about $w=0$:

$$\zeta_w(w) = \sum_{n=-\infty}^{\infty} C_n w^n.$$

Then we have
(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = x^2+y^2$,
(iii) $C_{2n-1} = -4\sqrt{ab}y$ for $n = 1,2,3,\cdots$.
(iv) $C_{2n} = -2((b-a)x - 2ab)$ for $n = 1,2,3,\cdots$.

Therefore the figures represented by $C_n = 0$ coincide with the figures represented by $C_n = 0$ in 7.3.

7.7.3. *Case 3.* We now consider the singular case $z = 1$ using the Laurent expansion of $\zeta_z(x, y)$ about $z = 1$:

$$\zeta_z(x, y) = \sum_{n=-\infty}^{\infty} C_n(z-1)^n.$$

Then we get
(i) $\cdots = C_{-4} = C_{-3} = C_{-2} = 0$, (ii) $C_{-1} = (a-b)x - 2\sqrt{ab}y + 2ab$,
(iii) $C_0 = \left(x - \dfrac{a-b}{4}\right)^2 + \left(y - \dfrac{\sqrt{ab}}{2}\right)^2 - \left(\dfrac{\sqrt{a^2+18ab+b^2}}{4}\right)^2$,
(iv) $C_n = \left(\dfrac{-1}{2}\right)^{n+1}((a-b)x + 2\sqrt{ab}y + 2ab)$ for $n = 1, 2, 3, \cdots$.
Therefore the figures represented by $C_n = 0$ also coincide with the figures represented by $C_n = 0$ in subsection 7.5. This case was considered in [25].

8. Skewed arbelos

We fix the two circles α and β and consider the valuable circle ζ_z touching α and β in the same sense represented by (7.2). The lines $\gamma_{\pm d}$ are denoted by $\zeta_{\pm 1}$. In this section we consider two special circles in the case $z = \pm 1$ by division by zero and division by zero calculus. However for the sake of simplicity, we confine ourself to the case $|z| \leq 1$, and consider the special circles in the case $z = 1$. See [21] for the case $|z| > 1$.

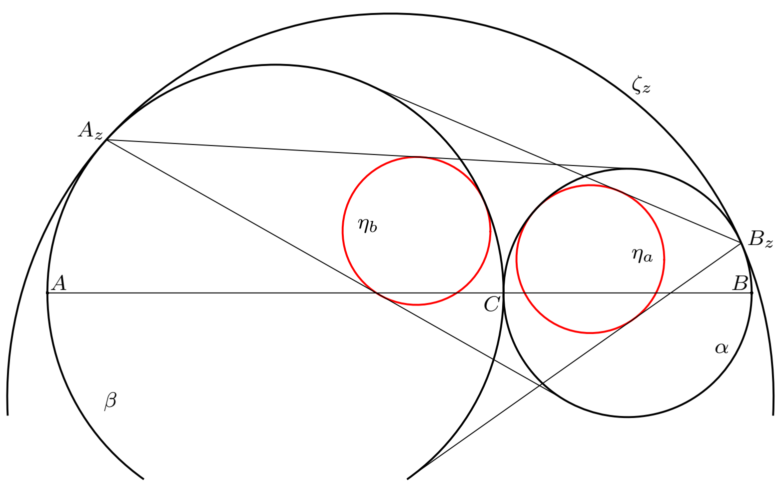

Figure 27.

Let B_z be the point of tangency of the circles α and ζ_z (see Figure 27). Let η_a be the circle touching α internally and the tangents of β from B_z. The point A_z and the circle η_b are defined similarly. The circles η_a and η_b are congruent and have common radius

$$r_\eta = |1 - z^2| r_{\mathrm{A}} \tag{8.1}$$

and have centers of coordinates

$$\left((1+z^2)r_{\rm A}, 2zr_{\rm A}\sqrt{\frac{a}{b}}\right) \ \ and \ \ \left(-(1+z^2)r_{\rm A}, 2zr_{\rm A}\sqrt{\frac{b}{a}}\right), \tag{8.2}$$

respectively [21], where recall $r_{\rm A} = ab/c$.

Let r_ζ be the radius of the circles ζ_z. Since $r_\zeta = c/|z^2 - 1|$ by (7.2), we have

$$r_\eta r_\zeta = ab \ \ if \ \ z \neq \pm 1.$$

If $z = 1$, then the circle ζ_z coincides with the line ζ_1, and the circles η_a and η_b and the points B_z and A_z coincide with the points B_1 and A_1, respectively. Therefore we get $r_\zeta = r_\eta = 0$ by Proposition 1.1 (see Figure 28). We also have the same relation in the case $z = -1$. Therefore in any case we have:

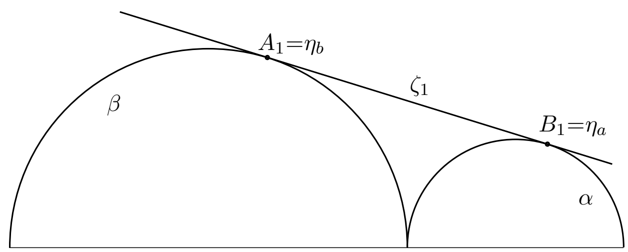

Figure 28: $z = 1$.

Theorem 8.1. $r_\eta = \dfrac{ab}{r_\zeta}$ *holds for any real number* z.

We now consider the circle η_a in the case $z = 1$ by division by zero calculus. By (8.1) and (8.2), the circle is represented by the equation

$$\eta_a(x,y) = \left(x - (1+z^2)r_{\rm A}\right)^2 + \left(y - 2zr_{\rm A}\sqrt{\frac{a}{b}}\right)^2 - \left((1-z^2)r_{\rm A}\right)^2 = 0.$$

We consider the Laurent expansion of $\eta_a(x,y)$ about $z = 1$:

$$\eta_a(x,y) = \sum_{n=-\infty}^{\infty} C_n (z-1)^n.$$

Then we have

(i) $\cdots = C_{-3} = C_{-2} = C_{-1} = 0$, (ii) $C_0 = (x - 2r_{\rm A})^2 + \left(y - 2r_{\rm A}\sqrt{a/b}\right)^2$,
(iii) $C_1 = -4r_{\rm A}((x-2a) + \sqrt{a/b}y)$, (iv) $C_2 = -2r_{\rm A}(x - 2a)$,
(v) $C_3 = C_4 = C_5 = \cdots = 0$.

Therefore $C_0 = 0$ represents the point of coordinates $(2r_{\rm A}, 2r_{\rm A}\sqrt{a/b})$, which coincides with the point B_1 (see Figure 29). $C_1 = 0$ represents the line BB_1. And $C_2 = 0$ represents the tangent of α at B. The three figures obtained by $C_n = 0$ $(n = 0, 1, 2)$ are described in Figure 29 in red.

The line BB_1 passes through the point of intersection of the axis and the circle $\zeta_0 = \gamma$, and the line meets the tangent of α at B, α, ζ_0, ζ_1 and the axis at the same angle, whose cosine equals $\sqrt{b/c}$. Similar results are also obtained by considering the circle η_b, and the resulting figures are described in Figure 29 in yellow. Since each of the three figures coincides with a line or a point, we get $r_\eta = 0$. Therefore Theorem 8.1 is also true.

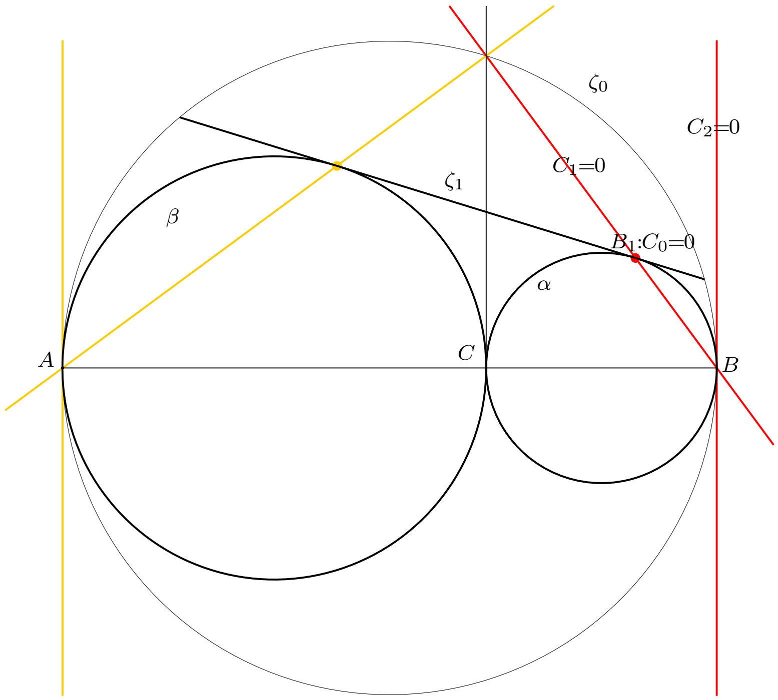

Figure 29: η_a and η_b are denoted by red and yellow, respectively.

9. The arbelos with overhang

In this section we consider another generalized arbelos called the arbelos with overhang [20]. Let A_h (resp. B_h) be a point on the half line CA (resp. CB) with initial point C such that $|A_hC| = 2(b+h)$ (resp. $|B_hC| = 2(a+h)$) for a real number h satisfying $-min(a,b) < h$. In [20] we have considered a generalized arbelos consisting of the three semicircles α_h, β_h and γ of diameters B_hC, A_hC and AB, respectively, constructed on the same side of AB. The figure is denoted by $(\alpha_h, \beta_h, \gamma)$ and is called the arbelos with overhang h (see Figures 30 and 31). The usual arbelos is obtained from $(\alpha_h, \beta_h, \gamma)$ if $h = 0$. The semicircles of diameters BC and AC constructed on the same side of AB as γ are denoted by α and β, respectively. We use a rectangular coordinate system with origin C such that the farthest point on α from AB has coordinates (a, a).

Assume $h \geq 0$, i.e., γ has a point in common with α_h and β_h. We define several touching circles for $(\alpha_h, \beta_h, \gamma)$ (see Figure 32): The incircle of the curvilinear triangle made by α, γ and the axis is denote by ε^a, i.e., ε^a is one of the twin circles of Archimedes of the arbelos formed by α, β and γ. The incircle of the curvilinear triangle made by α_h, γ and the axis is denote by ε_0^a. The incircle of the curvilinear triangle made by α, α_h and the radical axis of α_h and γ is denote by ε_1^a. The incircle of the curvilinear triangle made by α, α_h and γ is denoted by ε_2^a. The circle touching both α and γ externally and the axis from the side opposite to B is denote by ε_3^a. The circles ε^b and ε_i^b $(i = 0, 1, 2, 3)$ are defined similarly. Recall $r_A = ab/c$.

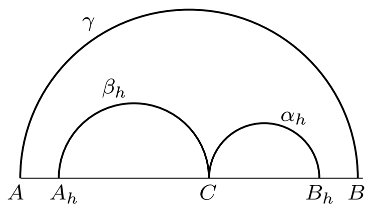

Figure 30: $h < 0$.

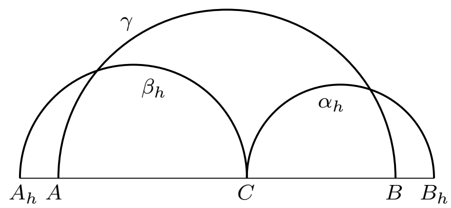

Figure 31: $0 < h$.

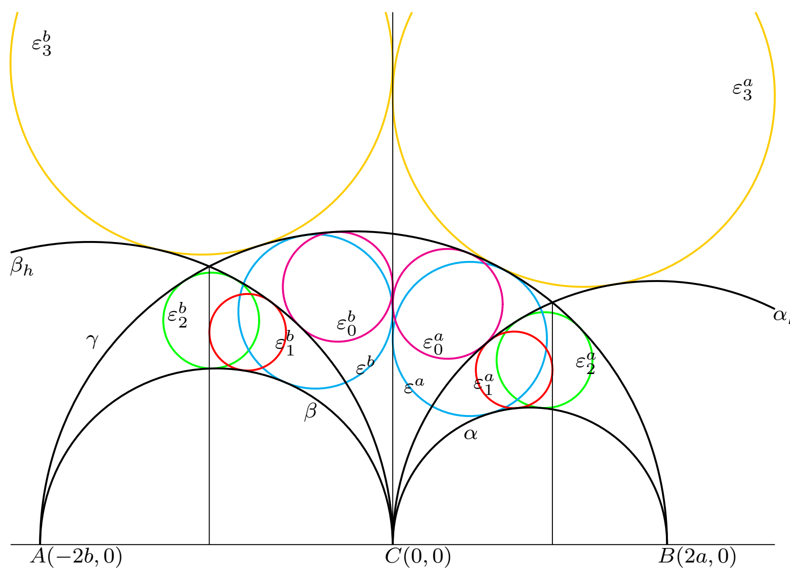

Figure 32.

Theorem 9.1 ([20]). *The following statements hold.*
(i) *The circles ε_0^a and ε_0^b have the same radius*

$$e_0 = \frac{ab}{a+b+h}.$$

(ii) *The circles ε_1^a and ε_1^b have the same radius*

$$e_1 = \left(\frac{1}{a} + \frac{1}{b} + \frac{h}{ab} + \frac{1}{h}\right)^{-1} = \left(\frac{1}{e_0} + \frac{1}{h}\right)^{-1}.$$

(iii) *The circles ε_2^a and ε_2^b have the same radius*

$$e_2 = \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{h}\right)^{-1} = \left(\frac{1}{r_A} + \frac{1}{h}\right)^{-1}.$$

(iv) *The circles ε_3^a and ε_3^b have the same radius*

$$e_3 = \frac{ab}{h}.$$

9.1. Division by zero. We consider the case $h = 0$ for the three pairs of congruent circles ε_i^a and ε_i^b $(i = 1, 2, 3)$ and ε^a and ε^b by division by zero. By Theorem 9.1, we get

$$e_1^{-1} = e_2^{-1} + e_3^{-1}. \tag{9.1}$$

If $h = 0$, then B_h and B coincide, and ε_1^a and ε_2^a also coincide with B, while ε_3^a coincides with the tangent of γ at B (see Figure 33). Similarly, ε_1^b and ε_2^b coincide with A, and ε_3^b coincides with the tangent of γ at A. Hence we have $e_1 = e_2 = e_3 = 0$ by Proposition 1.1. Hence (9.1) holds in this case.

We also get the following relation by Theorem 9.1:

$$e_3^{-1} + r_{\mathrm{A}}^{-1} = e_0^{-1}. \tag{9.2}$$

Assume $h = 0$. Then we get $e_3 = 0$ as just we have seen. While the circles ε^a and ε^b coincide with the circles ε_0^a and ε_0^b, respectively (see Figure 33). Hence we get $e_3 = 0$, while $r_{\mathrm{A}} = e_0$. Therefore (9.2) is true in this case.

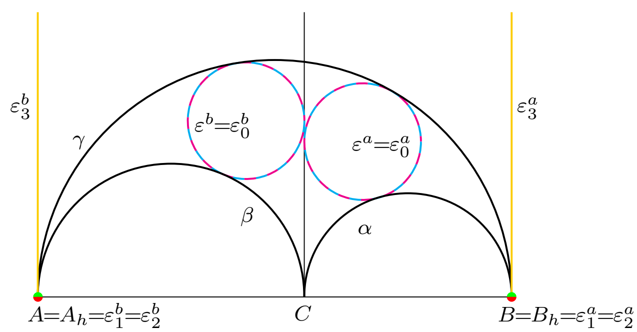

Figure 33: The case $h = 0$

9.2. The circles ε_1^a and ε_1^b by division by zero calculus. We consider the circles ε_i^a and ε_i^b $(i = 1, 2)$ in the case $h = 0$ by division by zero calculus. Firstly we consider the circles ε_1^a and ε_1^b. Let (x_1^a, y_1^a) be the coordinates of the center of ε_1^a. The radical axis of α_h and γ has the equation $x = a_x = 2ab/(b + h)$ and $x_1^a = a_x - e_1$ holds. With this relation and $(x_1^a - a)^2 + (y_1^a)^2 = (e_1 + a)^2$ and Theorem 9.1(ii), we get

$$(x_1^a, y_1^a) = \left(\frac{ab(2a + h)}{(a + h)(b + h)}, \frac{2a}{b + h}\sqrt{\frac{bh(c + h)}{a + h}}\right).$$

Therefore we get the equation $\varepsilon_1^a(x, y) = (x - x_1^a)^2 + (y - y_1^a)^2 - (e_1)^2 = 0$ representing the circle ε_1^a in terms of a, b and h. Then we have

$$\varepsilon_1^a(x, y) = \sum_{n=-\infty}^{\infty} C_n h^n = ((x - 2a)^2 + y^2)+$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2(2a^{n+1} - (a^n + (a^n + a^{n-1}b + a^{n-2}b^2 + \cdots + b^n))x)}{a^{n-1}b^n} h^n.$$

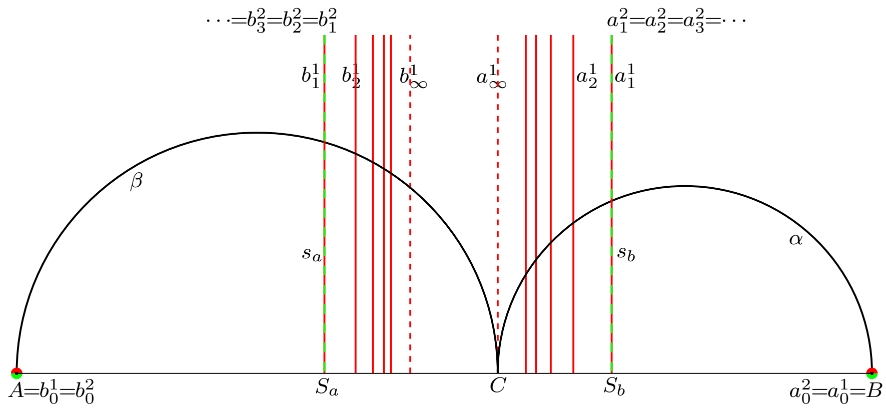

Figure 34: $a < b$.

We denote the figure represented by $C_n = 0$ by a_n^1. Then $a_0^1 = B$. For $n = 1, 2, \cdots$, the figure a_n^1 coincides with the line represented by the equation

$$x = \frac{2a}{n+2} \quad if \ a = b,$$

$$x = \frac{2a(a-b)}{2a - (1 + (b/a)^n)b} \quad if \ a \neq b.$$

From the last two equations, we see that a_1^1 coincides with the line s_b. If $a \leq b$ (resp. $b < a$), the line a_n^1 approaches to the axis (resp. the line represented by $x = 2a(a-b)/(2a-b)$, when n increases, which is denoted by a_∞^1 (see Figure 34).

For the circle ε_1^b, we get the equation $\varepsilon_1^b(x, y) = (x - x_1^b)^2 + (y - y_1^b)^2 - (e_1)^2 = 0$ representing the circle ε_1^b in a similar way, where

$$(x_1^b, y_1^b) = \left(-\frac{ab(2b+h)}{(a+h)(b+h)}, \frac{2b}{a+h}\sqrt{\frac{ah(c+h)}{b+h}} \right).$$

Therefore we get

$$\varepsilon_1^b(x, y) = \sum_{n=-\infty}^{\infty} C_n h^n = ((x+2b)^2 + y^2) +$$

$$\sum_{n=1}^{\infty} \frac{(-1)^n 2(2b^{n+1} + (b^n + (b^n + b^{n-1}a + b^{n-2}a^2 + \cdots + a^n))n)}{a^n b^{n-1}} h^n.$$

We denote the figure represented by $C_n = 0$ by b_n^1. Then $b_0^1 = A$, and b_n^1 coincides with the line represented by the equation

$$x = -\frac{2b}{n+2} \quad if \ a = b,$$

$$x = -\frac{2b(b-a)}{2b - (1 + (a/b)^n)a} \quad if \ a \neq b$$

for $n = 1, 2, \cdots$. From the two equations with (6.5), we see that the line b_1^1 coincides with the line s_a. If $b \le a$ (resp. $a < b$), the line b_n^1 approaches to the axis (resp. the line represented by the equation $x = -2b(b-a)/(2b-a)$), when n increases, which is denoted by b_∞^1. The figures a_n^1, b_n^1, a_∞^1 and b_∞^1 are described in Figure 34 in red, where a_n^2 and b_n^2 will be explained later.

9.3. The circles ε_2^a and ε_2^b by division by zero calculus. We consider the circles ε_2^a and ε_2^b. Let (x_2^a, y_2^a) be the coordinates of the center of ε_2^a. Solving the equations $(x_2^a - a)^2 + (y_2^a)^2 = (a + e_2)^2$ and $(x_2^a - (a + h))^2 + (y_2^a)^2 = (a + h - e_2)^2$ for x_2^a and y_2^a with Theorem 9.1(iii), we have

$$(x_2^a, y_2^a) = \left(\frac{ab(2a+h)}{ab+ch}, \frac{2a\sqrt{bch(a+h)}}{ab+ch}\right).$$

Therefore we get the equation $\varepsilon_2^a(x, y) = (x - x_2^a)^2 + (y - y_2^a)^2 - (e_2)^2 = 0$ representing the circle ε_2^a. Considering $\varepsilon_2^a(x, y)$ as a function of h, we have

$$\varepsilon_2^a(x, y) = \sum_{n=-\infty}^{\infty} C_n h^n = ((x-2a)^2 + y^2) + \sum_{n=1}^{\infty} (-1)^n \frac{2c^{n-1}(2a^2 - (c+a)x)}{a^{n-1}b^n} h^n.$$

We denote the figure represented by $C_n = 0$ by a_n^2. Then $a_0^2 = B$ and $a_1^2 = a_2^2 = \cdots$ coincides with the line s_b by (6.5), which coincides with the line a_1^1.

For the circle ε_2^b, we similarly get the equation $\varepsilon_2^b(x, y) = (x - x_2^b)^2 + (y - y_2^b)^2 - (e_2)^2 = 0$ representing the circle ε_2^b, where

$$(x_2^b, y_2^b) = \left(-\frac{ab(2b+h)}{ab+ch}, \frac{2b\sqrt{ach(b+h)}}{ab+ch}\right).$$

Therefore we get

$$\varepsilon_2^b(x, y) = \sum_{n=-\infty}^{\infty} C_n h^n = ((x+2b)^2 + y^2) + \sum_{n=1}^{\infty} (-1)^n \frac{2c^{n-1}(2b^2 + (b+c)x)}{a^n b^{n-1}} h^n.$$

We denote the figure represented by $C_n = 0$ by b_n^2. Then $b_0^2 = A$, and $b_1^2 = b_2^2 = b_3^2 = \cdots$ coincides with the line s_a by (6.5), which also coincides with the line b_1^1. The figures a_n^2 and b_n^2 are described in green in Figure 34.

10. Centers of similitude of two circles revisited

Considering the unexpected figures derived from division by zero calculus for the arbelos, we see the importances of the centers of similitude of two circles forming the arbelos, some of which are demonstrated in subsection 6.3 and sections 7 and 9. However we do not consider the points S_a and S_b and the lines s_a and s_b in detail, while it seems that they have never been considered before on the study of the arbelos. In this section we consider S_a and s_a in detail. The proofs are straightforward with algebraic manipulation and are omitted.

Recall that α_z is the circle touching β and γ in the opposite sense represented by (6.3). The coordinates of the center of the circle $\alpha_{\sqrt{bc}/a}$ and its radius are given by $((a-2b)/2, \sqrt{bc})$ and $a/2$ by Theorem 6.2. Let K be the point of tangency of α and the Archimedean circle touching α and γ in the opposite sense, which is denoted by $\beta_{\sqrt{c/b}}$ as stated after the proof of Theorem 6.2. We assume that the circle $\alpha_{\sqrt{bc}/a}$ touches β and γ at points L and M, respectively (see Figure 35). The points K, L and M have the following coordinates

$$K : \left(\frac{2ab}{b+c}, \frac{2a\sqrt{bc}}{b+c}\right), \quad L : \left(-\frac{2b^2}{b+c}, \frac{2b\sqrt{bc}}{b+c}\right), \quad M : \left(-\frac{2b^2}{b+c}, \frac{2c\sqrt{bc}}{b+c}\right).$$

Let N be the point of intersection of the lines AL and BM. Then N has coordinates $(a-b, \sqrt{bc})$. The line joining the farthest point on β from AB in the region $y < 0$ and the farthest point on γ from AB in the region $y > 0$ passes through the point S_a. We denote this line by v_a. Recall that σ_a is the circle of center S_a passing through A. The next theorem shows that the point S_a and the line s_a has many notable properties (see Figure 35 for the statements from (i) to (v) and see Figure 36 for the statements from (vi) to (viii)).

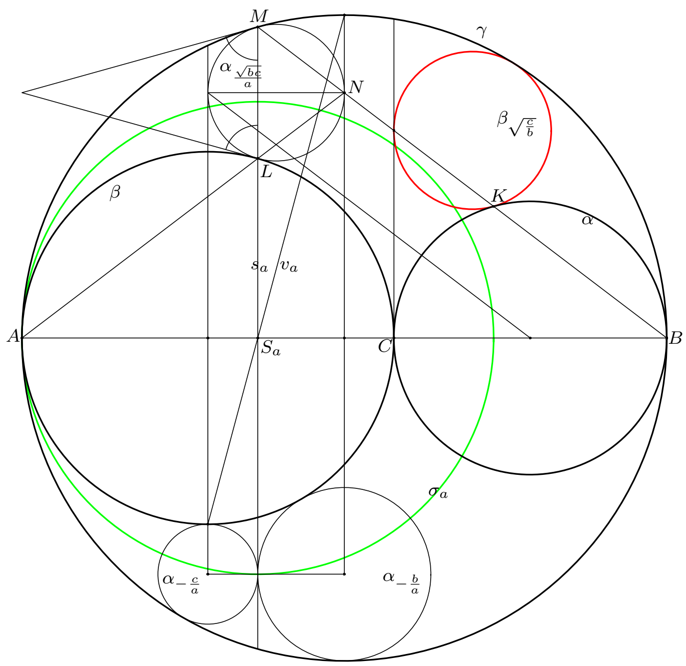

Figure 35.

Theorem 10.1. *The following statements are true.*
(i) *The line s_a passes through the points L and M.*
(ii) *The circumcircle of the triangle LMN coincides with the circle $\alpha_{\sqrt{bc}/a}$ and touches the perpendicular to AB at the center of γ at the point N.*
(iii) *The points B, K, M and N are collinear.*
(iv) *The circles $\alpha_{\pm b/a}$ and $\alpha_{\pm c/a}$ and s_a touch at one of the farthest points on σ_a from AB.*
(v) *The perpendicular to AB at the center of β (resp. γ) passes through the center of $\alpha_{\pm c/a}$ (resp. $\alpha_{\pm b/a}$) and touches $\alpha_{\pm\sqrt{bc}/a}$.*

(vi) *The perpendicular from K to AB touches the circle σ_a at the point of intersection of σ_a and AB.*
(vii) *If $v = \sqrt{(b^2+c^2)/(2a^2)}$, then the circles $\alpha_{\pm v}$ have center on the line s_a and the line v_a coincides with one of the internal common tangents of the two circles.*
(viii) *The circle touching σ_a internally at the point of intersection of α and σ_a in the region $y > 0$ and touching AB is Archimedean and touches AB at the point C. The radical axis of this circle and the circle $\beta_{\sqrt{\frac{c}{b}}}$ passes through the point B.*

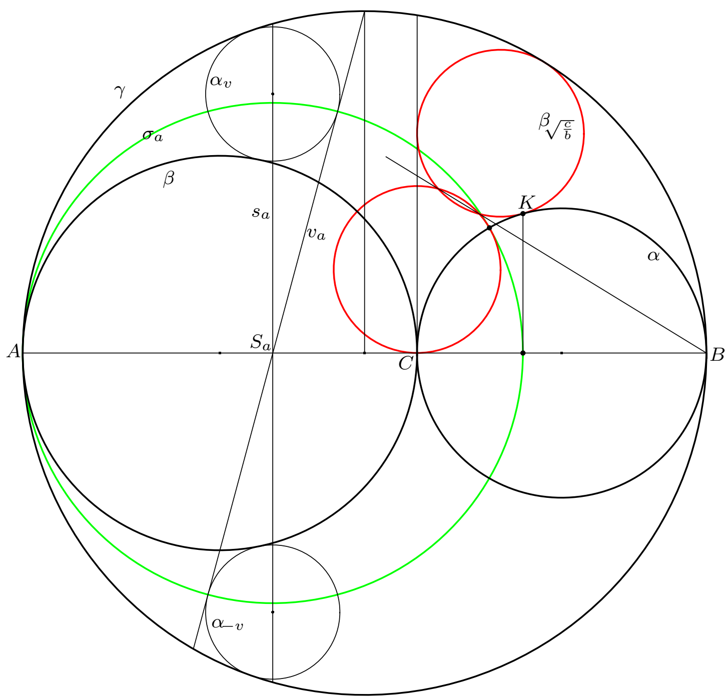

Figure 36.

The Archimedean circle described in (viii) is the Bankoff triplet circle [1]. Recall that $\Sigma^{\pm}$ are the points of intersection of the circles σ_a, σ_b and σ_c, whose coordinates are given by (6.7) (see Figure 37). The next theorem shows that the circle α_z touching the Bankoff triplet circle is uniquely determined independently of a and b for a positive or negative real number z.

Theorem 10.2. *The following statements are true.*
(i) *The circles $\alpha_{\frac{1+\sqrt{3}}{2}}$ and $\beta_{\frac{1+\sqrt{3}}{2}}$ touch the Bankoff triplet circle externally and their points of intersection lie on the circle σ_c, one of which coincides with Σ^+.*
(ii) *The circles $\alpha_{\frac{1-\sqrt{3}}{2}}$ and $\beta_{\frac{1-\sqrt{3}}{2}}$ touch the Bankoff triplet circle externally and their points of intersection lie on the circle σ_c, one of which coincides with Σ^-.*

The coordinates of the remaining point of intersection of $\alpha_{\frac{1+\sqrt{3}}{2}}$ and $\beta_{\frac{1+\sqrt{3}}{2}}$ are

$$\left(\frac{\left(6\sqrt{3}+11\right)ab(b-a)}{13a^2+3\left(5-2\sqrt{3}\right)ab+13b^2}, \frac{\left(9\sqrt{3}+10\right)abc}{13a^2+3\left(5-2\sqrt{3}\right)ab+13b^2}\right).$$

The coordinates of the remaining point of intersection of $\alpha_{\frac{1-\sqrt{3}}{2}}$ and $\beta_{\frac{1-\sqrt{3}}{2}}$ are

$$\left(\frac{\left(6\sqrt{3}-11\right)ab(a-b)}{13a^2+3\left(5+2\sqrt{3}\right)ab+13b^2}, -\frac{\left(9\sqrt{3}-10\right)abc}{13a^2+3\left(5+2\sqrt{3}\right)ab+13b^2}\right).$$

Since γ and the Bankoff triplet circle are orthogonal to σ_c, they are fixed by the inversion in σ_c. This implies that the two circles $\alpha_{\frac{1+\sqrt{3}}{2}}$ and $\beta_{\frac{1+\sqrt{3}}{2}}$ are interchanged by the inversion. Therefore the two circles are the inverse to each other by the inversion in σ_c. Similarly the circles $\alpha_{\frac{1-\sqrt{3}}{2}}$ and $\beta_{\frac{1-\sqrt{3}}{2}}$ are the inverse to each other by the inversion in σ_c.

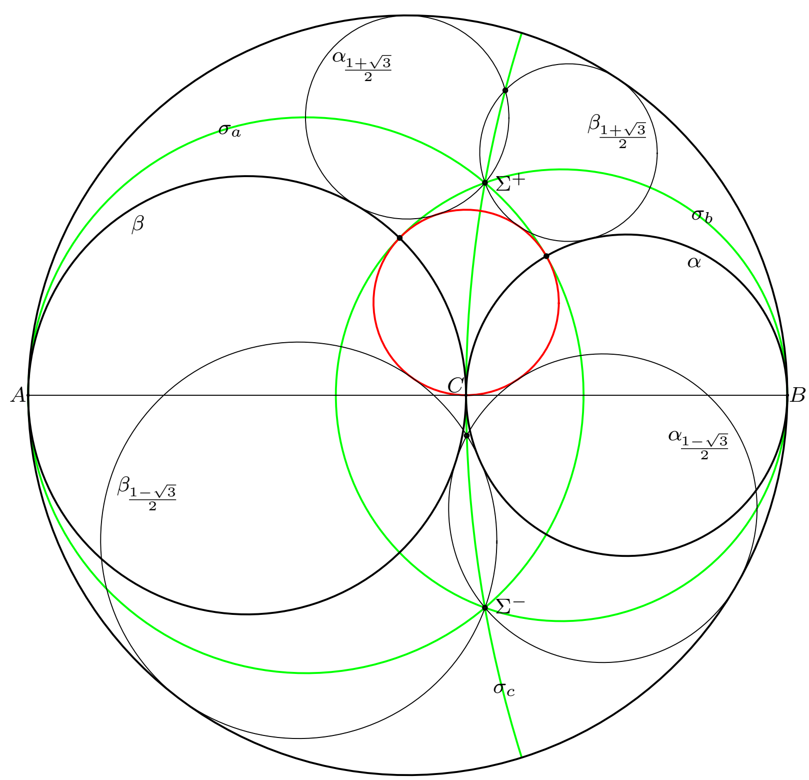

Figure 37.

11. Conclusion

We have shown that division by zero calculus gives us interesting and meaningful results in both singular case and non-singular case. We have seen that the unexpected figures such as the circle $\overline{\gamma}$ and the line s_a are one of the most important and essential ones for the study on the arbelos, but those figures have finally got attention through the study using division by zero calculus.

Mathematicians usually may think that Laurent expansion belongs to analysis, but it seems that division by zero calculus using Laurent expansion is a very powerful tool even for the study of geometry. However we have no idea why we can get such notable figures by division by zero calculus at this time of writing. Thereby we hope that many mathematicians will join the study of division by zero calculus and will get the reason for this and also find huge number of marvelous things derived from division by zero calculus.

For more results brought by division by zero and division by zero calculus in geometry see [2], [10], [11], [12], [13], [14], [15], [16], [17], [22], [23], [24], [25], where all the papers except [14], [24] and [25] are considering problems in Wasan geometry (Japanese geometry) or related to this geometry.

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