International Journal of Reproducing Kernels
Vol. 2 No. 1 (2023)

REPRESENTATIONS OF QUOTIENTS $g/f = h$ IN TERMS OF $g = fh$

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ABSTRACT. In this paper, we shall consider quotients g/f of general functions f and g in some deep and natural meanings, in a natural setting. The general theory of reproducing kernels will give the natural theory of the problem. In particular, we will consider the division by any functions containing the division by zero. We wish to know the meaning of the function g/f when the function f has zero points, with the definition of g/f.

1. Introduction

In this paper, we shall consider quotients g/f of general functions f and g in some deep and natural meanings, in a natural setting. The general theory of reproducing kernels will give the natural theory of the problem. In particular, we will consider the division by any functions containing the division by zero. We wish to know the meaning of the function g/f when the function f has zero points, with the definition of g/f.

First, what is a function $y = f(x)$ on (a, b)? For even the typical functions $L_2(a, b)$ having a good looking, we will not be able to get the functions as the corresponding (mapping) from the points on (a, b) to some space on $\mathbf{R}$ or $\mathbf{C}$, indeed the points are too many to consider them. This question will be more clear when we consider the inversions $1/f$ of the functions f of $L_2(a, b)$. Therefore, we will realize a function as a member of some function space, and the function space represents the functions as a global property over each point value.

The general theory of reproducing kernels will give the natural theory of the problem for the quotients g/f of general functions f and g in some deep and natural meanings.

This paper will have the natures of addition on the general paper [5] for the reproducing kernel theory and of an extension of the papers [2, 4] and is impacted by the recent norm inequalities by A. Yamada [20]. The paper will also represent a very interesting nature of the theory of reproducing kernels such that for **an arbitrary mapping** we considered some representation of its inversion [12] and as its great extensions we obtained the explicit representation [3] of the implicit functions in the theorem of implicit function existence.

2000 *Mathematics Subject Classification.* Primary 30C40, 44A05; Secondary 46E22,45A05.
Key words and phrases. General quotient of two functions, division by any function, reproducing kernel, Tikhonov regularization, norm inequality, product of reproducing kernels, division by zero, multiplicative operator, Moore-Penrose generalized solution, Sobolev Hilbert space.

Furthermore, by using the theory of reproducing kernels, we can consider differential and integral equations with variable coefficients that are **arbitrary functions**, based on **general discretization principle and back ward transformation method**. See [14], pages, 131, 147-148.

2. Quotients of Two Functions

Now, for some general two functions f, g on a set E, we will consider the quotient

$$\frac{g}{f}. \tag{2.1}$$

In order to consider such a function (2.1), we shall consider the related equation

$$f_1(p)f(p) = g(p) \quad \text{on} \quad E \tag{2.2}$$

for some function f_1 on the set E. If the solution f_1 in (2.2) on the set E exists, then the solution f_1 will give the meaning of the fractional (quotient) function (2.1). So, the problem may be transformed to the very general and popular equation (2.2).

Here, the serious problem is the case of the zero points of the function f; because we can not give the meaning of the function (2.1) there, intuitively. However, except the zero points of f, the solution f_1 gives the quotient (2.1) point wisely.

In this starting point, the function f is initially given. So, for analyzing the equation (2.2), we must introduce a suitable function space containing the function f_1 and then we find the induced function space containing the product $f_1 \cdot f$.

This idea means that the function g has a natural restriction, because it is the product of the function f and f_1 of some general function space. (*Children are intrinsically influenced by their mothers. In fact, in Japanese, for g/f, f and g are related to mother and child, respectively. f is the first and g is the next.*)

Then, we will be able to consider the solution of the equation (2.2). Here, on this line, we will show that we can discuss the above problem in a very general setting. Indeed, this may be considered for an arbitrary function f on the set E that is non-identically zero on the set E by using the theory of reproducing kernels.

At first, we note that for an arbitrary function $f(p)$, there exist many reproducing kernel Hilbert spaces containing the function $f(p)$; the simplest reproducing kernel is given by $f(p) \times \overline{f(q)}$ on $E \times E$. In general, a reproducing kernel Hilbert space $H_K(E)$ on E admitting a reproducing kernel K on $E \times E$ is uniquley determined by a positive definite Hermitian form (kernel; now $f(p) \times \overline{f(q)}$) and the space is characterized by the very natural property that any point evaluation $f(p)$ is a bounded linear operator on $H_K(E)$ for any point $p \in E$. Therefore, secondary, we shall consider such a reproducing kernel Hilbert space $H_{K_1}(E)$ admitting a reproducing kernel K_1 containing the functions $f_1(p)$.

Then, we note the very interesting fact that the products $f_1 \cdot f$ determine a natural reproducing kernel Hilbert space that is induced by $H_{K_1}(E)$ and $H_K(E)$. In fact, the space in question is the reproducing kernel Hilbert space $H_{K_1K}(E)$ that is determined by the product $K_1 \cdot K$ and, furthermore, we obtain the fundamental and beautiful norm inequality

$$\|f_1 f\|_{H_{K_1 K}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{2.3}$$

This important inequality (2.3) means that for the linear operator $\varphi_f(f_1)$ on $H_{K_1}(E)$ (for a fixed function f), defined by

$$\varphi_f(f_1)(p) \equiv f_1(p) f(p), \tag{2.4}$$

we obtain the inequality

$$\|\varphi_f(f_1)\|_{H_{K_1 K}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f\|_{H_K(E)}. \tag{2.5}$$

This means that the mapping (multiplicative operator) φ_f is a bounded linear operator from $H_{K_1}(E)$ into $H_{K_1 K}(E)$. See [14], Tensor Product of Reproducing Kernels; pages 105-109.

3. Moore-Penrose Generalized Solution

As the very natural solution of the operator equation (2.2), we will consider the best approximation, for any function g of the space $H_{K_1 K}(E)$

$$\inf_{f_1 \in H_{K_1}(E)} \left\{ \|\varphi_f(f_1) - g\|_{H_{K_1 K}(E)}{}^2 \right\}, \tag{3.1}$$

that leads to the Moore-Penrose generalized solution of (2.2).

So, simply we will recall the essential and general properties of the best approximation from [14], pages 166-169.

Let L be any bounded linear operator from a reproducing kernel Hilbert space $H_K(E)$ admitting a kernel $K : E \times E \to \mathbb{C}$ into a Hilbert space $\mathcal{H}$. We set $K_p = K(\cdot, p)$.

For any member $\mathbf{d}$ of $\mathcal{H}$, we will consider the best approximation problem

$$\inf_{f \in H_K(E)} \|Lf - \mathbf{d}\|_{\mathcal{H}}. \tag{3.2}$$

Set

$$k(p,q) \equiv \langle L^* L K_q, L^* L K_p \rangle_{H_K(E)} = L^* L L^* L[K_q](p) \tag{3.3}$$

and

$$P = \mathrm{proj}_{H_K(E) \to \ker(L)^\perp} = \mathrm{proj}_{H_K(E) \to \overline{\mathrm{Ran}(L^* L)}}. \tag{3.4}$$

(Change $\mathcal{H}$ by $H_K(E)$).

Theorem A: *Under the notations* (3.3) *and* (3.4), *we have*

$$H_k(E) = \{L^* L f \, : \, f \in H_K(E)\} \tag{3.5}$$

and the inner product is given by

$$\langle L^* L f, L^* L g \rangle_{H_k(E)} = \langle P f, g \rangle_{H_K(E)} \tag{3.6}$$

for $f, g \in H_K(E)$.

Theorem B: *Equation* (3.2) *admits a solution if and only if* $L^* \mathbf{d} \in H_k(E)$. *If this is the case, then we have* $L^* \mathbf{d} = L^* L \tilde{f}$ *for some* $\tilde{f} \in H_K(E)$ *and* $\tilde{f}$ *is a solution to* (3.2).

Let $f_{\mathbf{d}} \in H_K(E)$ be the element such that

$$L^*\mathbf{d} = L^*Lf_{\mathbf{d}} \tag{3.7}$$

with $f_{\mathbf{d}} \in \ker(L)^{\perp}$.

The extremal function $f_{\mathbf{d}}(p)$ has the following representation:

Theorem C: *Keep to the same assumption as above. Then we have*

$$f_{\mathbf{d}}(p) = \langle L^*\mathbf{d}, L^*LK_p\rangle_{H_k(E)} \quad (p \in E). \tag{3.8}$$

The adjoint operator L^* of L, as we see from equality:

$$L^*\mathbf{d}(p) = \langle L^*\mathbf{d}, K_p\rangle_{H_K(E)} = \langle \mathbf{d}, LK_p\rangle_{\mathcal{H}} \quad (p \in E), \tag{3.9}$$

is represented by the known data $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From Theorems A, B, C, we see that the problem is well established by the theory of reproducing kernels. That is, the existence, the uniqueness and the representation of the solutions in the problem are well formulated. In particular, note that the adjoint operator is represented in a good way; this fact will turn out very important in our framework. The extremal function $f_{\mathbf{d}}$ is the **Moore-Penrose generalized inverse (solution)** $L^{\dagger}\mathbf{d}$ of the operator equation $Lf = \mathbf{d}$. The criteria in Theorem A is involved and the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is, in general, not good.

Furthermore, we note that

Theorem D: *The following are equivalent*:

(1) *L is injective*;
(2) *L^*L is injective*;
(3) *$\{L^*LK_x\}_{x\in E}$ is complete in $H_K(E)$*;
(4) *$L^*L : H_K(E) \to H_k(E)$ is isometry*

([14], page178).

In particular, note that even the simple case, L^* is still, in general, not injective, and so we can not say that from $L^*\mathbf{d} = L^*Lf$, $Lf = \mathbf{d}$, the classical solution.

Now we shall apply the above general theory to our case. The situation will be essentially simple.

At first,

$$\varphi_f{}^*(g)(p) = (g, \varphi_f K_1(\cdot,p))_{H_{K_1K_2}(E)} = (g, f(\cdot)K_1(\cdot,p))_{H_{K_1K_2}(E)}$$

and the existence of the best approximations is that g is represented in the product for some function f_1

$$g = f_1 f$$

and the best approximation function is f_1 itself. In the present case,

$$H_k = \{\varphi_f{}^*\varphi_f f_1; f_1 \in H_{K_1}(E)\}.$$

For the multiplicative operator $\varphi_f f_1 = f_1 f$, **practically we can assume that it is injective**. Furthermore, **we assume that the operator**

$$\varphi_f{}^*\varphi_f\varphi_f{}^*\varphi_f$$

is an identity on the space $H_{K_1}(E)$.

Then, the above theory is clear all and we can obtain the surprising desired result:

For the product $g = f_1 f; f_1 \in H_{K_1}(E), f \in H_K(E)$, we obtain the representation

$$f_1(p) = \Big(f(p_2) \left(g(p_1), f(p_1)K_1(p_1,p_2) \right)_{H_{K_1K}(E)}, f(p_2)K_1(p_2,p) \Big)_{H_{K_1K}(E)}. \quad (3.10)$$

Indeed, we can see directly the formula as follows:

The operator

$$\varphi_f{}^* \varphi_f \varphi_f{}^* \varphi_f$$

is an identity on the space $H_{K_1}(E)$, by the assumption. And so in the identity

$$f_1 = \varphi_f{}^* \varphi_f \varphi_f{}^* \varphi_f f_1$$

by setting

$$\varphi_f f_1 = f_1 f = g,$$

we obtain the desired result.

In particular, *when* $g = 1 \in H_{K_1K}(E)$, we have the representation

$$\frac{1}{f(p)} = \Big(f(p_2) \left(1, f(p_1)K_1(p_1,p_2) \right)_{H_{K_1K}(E)}, f(p_2)K_1(p_2,p) \Big)_{H_{K_1K}(E)}. \quad (3.11)$$

Quotient was represented by Product.

Trivial cases:

In order to see the nature of the formula and to check the results, we will examine the trivial cases.

For any functions f and f_1:

We consider the kernels

$$K(p,q) = f(p) \times \overline{f(q)}$$

and

$$K_1(p,q) = f_1(p) \times \overline{f_1(q)}.$$

Then, any member of $H_K(E)$ is expressible in the form $Cf(p)$ and

$$\|Cf(p)\|^2_{H_K(E)} = |C|^2.$$

For the space of $H_{K_1}(E)$, the structure is the similar. Therefore, any member $g \in H_{K_1K}(E)$ is expressible in the form $g(p) = CC_1 f(p) f_1(p)$ and

$$\|CC_1 f(p) f_1(p)\|^2_{H_{K_1K}(E)} = |C|^2 |C_1|^2 = \|Cf(p)\|^2_{H_K(E)} \|C_1 f_1(p)\|^2_{H_{K_1}(E)}$$

and we see that the formula is right.

For any functions f and for some function space for f_1:

For the functions f_1, we will consider a general reproducing kernel Hilbert space $H_{K_1}(E)$. Then, the space $H_{K_1K}(E)$ is expressible in the form $g(p) = Cf(p)F_1(p)$ with $F_1 \in H_{K_1}(E)$ and

$$\|Cf(p)F_1(p)\|^2_{H_{K_1K}(E)} = |C|^2 \|F_1\|^2_{H_{K_1}(E)}$$

and we see that the formula is right.

The solution f_1 for $g = f_1 f$ may be represented as a quotient

$$f_1 = \frac{g}{f}$$

in our natural sense.

Since our case, for the zero points of the function f, the function f_1 has the natural values, we will have a serious interest for the natural values. Since the function f_1 is a usual function, the zero of the function f is canceled, however, how will be its property? How will its behave?

In those formulas, in general, the representations may not be used analytically. Furthermore, the basic assumption of the identity is not, generally, valid. So, we will consider some more practical and general formulas, by the Tikhonov regularization.

4. By the Tikhonov Regularization

When the data contain error or noise in some practical cases, the exact theory by the Moore-Penrose generalized solutions is not applicable, therefore, we shall introduce the concept of the Tikhonov regularization with general data g.

We will consider the Tikhonov functional, for any $g \in H_{K_1K}(E)$ and for any positive α:

$$\inf_{f_1 \in H_{K_1}(E)} \left\{ \alpha \|f_1\|_{H_{K_1}(E)}{}^2 + \|\varphi_f(f_1) - g\|_{H_{K_1K}(E)}{}^2 \right\}. \tag{4.1}$$

The extremal function $f_{1,\alpha}$ exists uniquely and always. If the limit

$$\lim_{\alpha \downarrow 0} f_{1,\alpha}(p) \quad \text{on} \quad E$$

exists, then the limit function is the Moore-Penrose generalized solution for the equation (2.2) in the sense of the best minimum norm.

Now we recall the fundamental results for the Tikhonov functionals.

See [14], pages 193-196 and pages 179-186.

Theorem E: *Let $\alpha > 0$. For a bounded linear operator L for a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$, the following minimizing problem admits a unique solution*;

$$\min_{f \in H_K} \left(\alpha \, \|f\|_{H_K}{}^2 + \|\mathbf{d} - Lf\|_{\mathcal{H}}{}^2 \right). \tag{4.2}$$

Furthermore, the minimum is attained by

$$f_{\mathbf{d},\alpha} = (L^*L + \alpha)^{-1} L^* \mathbf{d} = \left(\int_{\mathbb{R}} \frac{1}{\lambda + \alpha} dE_\lambda \right) L^* \mathbf{d} \tag{4.3}$$

by using the spectral decomposition. Furthermore, $\mathbf{d} \mapsto f_{\mathbf{d},\alpha}$ *is almost the inverse of* L *in the following sense:*

$$\lim_{\alpha\downarrow 0} f_{Lg,\alpha} = g \tag{4.4}$$

in $H_K(E)$ *for all* $g \in H_K(E)$ *and when there exists the Moore-Penrose generalized solution,*

$$\lim_{\alpha\downarrow 0} Lf_{\mathbf{d},\alpha} = \mathbf{d} \tag{4.5}$$

in $\mathcal{H}$*.*

Theorem F: *Let* $L : H_K(E) \to \mathcal{H}$ *be a bounded linear operator. Then define an inner product*

$$\langle f_1, f_2\rangle_{H_{K_\lambda}(E)} = \alpha \langle f_1, f_2\rangle_{H_K} + \langle Lf_1, Lf_2\rangle_{\mathcal{H}} \tag{4.6}$$

for $f_1, f_2 \in H_K$*. Then* $(H_K, \langle\cdot,\cdot\rangle_{H_{K_\alpha}(E)})$ *is a reproducing kernel Hilbert space whose reproducing kernel is given by:*

$$K_\alpha(p,q) = [(\alpha + L^*L)^{-1}K_q](p). \tag{4.7}$$

Here, $K_\alpha(p,q)$ *satisfies*

$$K_\alpha(p,q) + \frac{1}{\alpha}\langle L[(K_\alpha)_q], L[K_p]\rangle_{\mathcal{H}} = \frac{1}{\alpha}K(p,q), \tag{4.8}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases.

Theorem G: *Under the same assumption as Theorems E and F,*

$$f \in H_K \mapsto \alpha \, \|f\|_{H_K(E)}{}^2 + \|Lf - \mathbf{d}\|_{\mathcal{H}}{}^2 \in \mathbb{R}$$

attains the minimum only at $f_{\mathbf{d},\alpha} \in H_K(E)$ *which satisifes*

$$f_{\mathbf{d},\alpha}(p) = \langle \mathbf{d}, L[(K_\alpha)_p]\rangle_{\mathcal{H}}. \tag{4.9}$$

Furthermore, $f_{\mathbf{d},\alpha}(p)$ *satisfies*

$$|f_{\mathbf{d},\alpha}(p)| \le \|L\|_{H_K(E)\to\mathcal{H}} \sqrt{\frac{K(p,p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{4.10}$$

Note that when it is involved to realize the reproducing kernel Hilbert space $H_{K_1K}(E)$ and when we can look a reproducing kernel Hilbert space $H_{\mathbf{K}}(E)$ admitting a reproducing kernel $\mathbf{K}$ satisfying

$$K_1(p,q)K(p,q) \ll \mathbf{K}(p,q) \quad (p,q) \in E^2$$

– the left minus the right is a positive definite quadratic form function – and its structure is simple, from the properties

$$H_{K_1K}(E) \subset H_{\mathbf{K}}(E)$$

and

$$\|g\|_{H_{K_1K}(E)} \ge \|g\|_{H_{\mathbf{K}}(E)}, \quad g \in H_{K_1K}(E),$$

we see that in the above theory we can use the space $H_{\mathbf{K}}(E)$ instead of the space $H_{K_1K}(E)$.

By the above general theorems, we can obtain the general results:

For any $g \in H_{K_1K}(E)$ *and for any* $\alpha > 0$*, the minimizing function function* $f_1^{(\alpha)} \in H_{K_1}(E)$ *of*

$$\inf_{f_1 \in H_{K_1}(E)} \left\{ \alpha \|f_1\|_{H_{K_1}(E)}{}^2 + \|\varphi_f(f_1) - g\|_{H_{K_1K}(E)}{}^2 \right\}, \tag{4.11}$$

among the functions $f_1 \in H_{K_1}(E)$ *is given by*

$$f_1^{(\alpha)}(p) = (g(\cdot), f(\cdot)K_\alpha(\cdot,p))_{H_{K_1K}(E)}. \tag{4.12}$$

Here, the reproducing kernel $K_\alpha(\cdot,p)$ *is determined by the functional equation*

$$K_\alpha(p,q) + \frac{1}{\alpha}(f(\cdot)K_\alpha(\cdot,q), f(\cdot)K_1(\cdot,p))_{H_{K_1K}(E)} = \frac{1}{\alpha}K_1(p,q). \tag{4.13}$$

The problem is the world of approximation and approximating spaces are clear and beautiful, and so there is no problem as in stated in the above general theorems. The representation (4.12) is not direct by using the solution of (4.13). However, the equation (4.13) is the Fredholm integral type in the second kind and so, the solutions are effective and numerically stable, as we see from the real inversion formula of the Laplace transform by taking a small α. See Chapter 4 of [14].

In particular, H. Fujiwara solved the integral equation corresponding to (4.13) for the real inversion formula of the Laplace transform with 6000 points discretization with **600 digits precision** based on the concept of **infinite precision**. Then, the regularization parameters were $\alpha = 10^{-100}, 10^{-400}$ surprisingly. H. Fujiwara was successful in deriving numerically the inversion for the Laplace transform of the distribution delta which was proposed by V. V. Kryzhniy as a difficult case. This fact will mean that the above results valid for very general functions approximated by the functions of the reproducing kernel Hilbert space.

5. Typical Examples

We shall give the typical examples.

5.1. Bergman and Szegö Kernels. For the Bergman kernel and the Szegö kernel on a regular domain D on the complex $z = x + iy$ plane, we have the basic and deep relation

$$K(z, \overline{u}) >> 4\pi \hat{K}(z, \overline{u})^2$$

which was given by D. A. Hejhal [22]. This profound result using the Riemann theta function was given on the long historical results as in

G.F.B. Riemann (1826-1866); F. Klein (1849-1925); S. Bergman; G. Szegö; Z. Nehari; M.M. Schiffer; P.R. Garabedian (1949 published); D.A. Hejhal (1972 published).

It seems that any elementary proof is impossible, however, the result will, in particular, mean a fairly simple and fundamental inequality:

For two functions φ and ψ of $H_2(D)$, analytic Hardy space, we obtain the generalized isoperimetric inequality

$$\frac{1}{\pi}\int\int_D |\varphi(z)\psi(z)|^2 dxdy \leq \frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2|dz| \frac{1}{2\pi}\int_{\partial D} |\psi(z)|^2|dz|, \tag{5.1}$$

and we can determine completely the case holding the equality here. In the thesis [42] of the author published in 1979 the result was given. The author realized the importance of the abstract and general theory of reproducing kernels by N. Aronszajn ([1]). In the paper, the core part was to determine the equality statement in the above inequality, surprisingly enough, some deep and general independing poof was appeared 26 years later in A. Yamada ([82]). A. Yamada was developed deeply equality problems for some general norm inequalities derived by the theory of reproducing kernels and it was published in the book appendix of [14]. Very recently his theory is developing much more in [19].

Experiments:

On the unit disc, the Szegö kernel is given by

$$K(z, \overline{u}) = \frac{1}{1 - \overline{u}z}$$

with the norm

$$\sqrt{\frac{1}{2\pi}\int_{\partial D} |\varphi(z)|^2|dz|}.$$

We shall consider the case: K_1 and K are the same Szegö kernel and so $H_{KK}(D)$ is the Bergman space on the unit disc.

By the formula (3.10), we obtain, for $m \geq 1$ and for

$$g = z^m, \quad f = z,$$

$$\frac{1}{(m+1)^2} z^{m-1}.$$

That is, the identity assumption is not valid.

Therefore, we shall use the formula (3.8).

First, $L^*LK(\cdot, \overline{u})$ is

$$L^*LK(z, \overline{u}) = \sum_{n=0}^{\infty} \frac{1}{n+2} z^n \overline{u}^n$$

and

$$k(z, \overline{u}) = L^*LL^*LK(z, \overline{u}) = \sum_{n=0}^{\infty} \frac{1}{(n+1)^2} z^n \overline{u}^n.$$

Therefore, the reproducing kernel Hilbert space H_k is realized as follows:

Any member $f \in H_k$ is expressible in the form

$$f(z) = \sum_{n=0}^{\infty} C_n \frac{1}{(n+1)^2} z^n$$

with

$$\sum_{n=0}^{\infty} |C_n|^2 \frac{1}{(n+1)^2} < \infty$$

and its norm is given by

$$||f||_{H_k}^2 = \sum_{n=0}^{\infty} |C_n|^2 \frac{1}{(n+1)^2}.$$

By using the formula (3.8) directly for $\mathbf{d} = z^m$ and $f = z$, we have the expected right result

$$f_{\mathbf{d}} = z^{m-1}.$$

Formally, we have the result even, for $m = 0$

$$f_{\mathbf{d}} z = 1$$

and

$$f_{\mathbf{d}} = \frac{1}{z}.$$

5.2. Sobolev Spaces. We have similar results and theory for the Sobolev spaces. For example, let ρ be a positive continuous function on (a, b) satisfying $\rho \in L_1(a, b)$. Let f_j be complex-valued functions on (a, b) satisfying $\lim_{x \to a-0} f_j(x) = 0$. Then, we have the inequality

$$\int_a^b |(f_1(x) f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right) \rho(x)}$$

$$\leq 2 \int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)} \int_a^b |f_2'(x)|^2 \frac{dx}{\rho(x)},$$

when the integrals in the last part are finite. Equality holds here if and only if each f_j is expressible in the form $C_j K_\rho(x, x_2)$ for some constants C_j and for some point $x_2 \in [a, b]$ which is independent of j. Here, $K_\rho(x, \cdot)$ is the reproducing kernel of the Sobolev space with the norm

$$\sqrt{\int_a^b |f_1'(x)|^2 \frac{dx}{\rho(x)}} < \infty$$

([8, 17]).

We will consider the first order Sobolev Hilbert spaces $H(a, b; \mathbf{R})$, $(a, b > 0)$, as the basic reproducing kernel Hilbert space with finite norms

$$\sqrt{\int_{\mathbf{R}} (a^2 |f'(x)|^2 + b^2 |f(x)|^2)\, dx}$$

admitting the reproducing kernel $K_{H(a,b;\mathbf{R})}(x, x_1)$

$$K_{H(a,b;\mathbf{R})}(x, x_1) = \frac{1}{2ab} \exp\left(-\frac{b}{a}|x - x_1|\right).$$

See [14], pages 10-18 for the related basic materials.

We will consider this space as in the Szegö space in (5.1). Note the identity

$$K_{H(a,b;\mathbf{R})}(x,x_1)^2 = \frac{1}{ab}\frac{1}{2a(2b)}\exp\left(-\frac{(2b)}{a}|x-x_1|\right)$$
$$= \frac{1}{ab}K_{H(a,2b;\mathbf{R})}(x,x_1).$$

From the construction of the norms admitting the reproducing kernels corresponding to the product and multiplication of a positive number for reproducing kernels, we obtain the norm inequality as in (5.1).

(A) For any $f, g \in H(a,b;\mathbf{R})$, we have the norm inequality

$$\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + 4b^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{1}{ab}\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + b^2|g(x)|^2\right) dx.$$

Of course, we have

(A') For any $f, g \in H(a,b/2;\mathbf{R})$, we have the norm inequality

$$\int_{\mathbf{R}} \left(a^2|(f(x)g(x))'|^2 + b^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{2}{ab}\int_{\mathbf{R}} \left(a^2|f'(x)|^2 + \frac{b^2}{4}|f(x)|^2\right) dx \int_{\mathbf{R}} \left(a^2|g'(x)|^2 + \frac{b^2}{4}|g(x)|^2\right) dx.$$

Finite Interval Cases

If we note that the kernel on an interval $[c,d], -\infty \leq c < d \leq +\infty$

$$K_{H(a,b;[c,d])}(x,x_1) = \frac{1}{2ab}\exp\left(-\frac{b}{a}|x-x_1|\right)$$

is the reproducing kernel on the Hilbert space $H(a,b;[c,d])$ with finite norms

$$\sqrt{\int_{[c,d]} (a^2|f'(x)|^2 + b^2|f(x)|^2)\, dx + ab(|f(c)|^2 + |f(d)|^2)} < \infty$$

as in the whole space case, the results in (A) are valid in the corresponding way. This fact for the norm may be confirmed directly by checking the reproducing property of the kernel as in [14], pages 11-12. Meanwhile, the kernel $K_{H(a,b;[c,d])}(x,x_1)$ is the restriction to the interval $[c,d]$ of the kernel $K_{H(a,b;\mathbf{R})}(x,x_1)$ and so by the general property of reproducing kernels, we see that any member $f(x)$ of $H(a,b;[c,d])$ is the restriction of a function $h(x)$ in $H(a,b;\mathbf{R})$ and its norm is given by

$$||f||_{H(a,b;[c,d])} = \min ||h||_{H(a,b;\mathbf{R})},$$

where the minimum is taken over all functions h in $H(a,b;\mathbf{R})$ satisfying

$$f(x) = h(x) \quad on \quad [c,d].$$

See [14], pages 78-80. In particular, note that any member $f(x)$ of $H(a,b;[c,d])$ has a good property on the interval $[c,d]$.

Generalizations For the First Order Sobolev Hilbert Spaces

From the products of different type kernels, we shall consider the corresponding norm inequalities as generalizations.

First recall the result [14], page 16-17:

For the half-open interval $I=[a,b)$, we consider a positive continuous function $\rho : I \to (0,\infty)$, such that

$$\rho \in L^1[a,x] \quad (x \in I) \tag{5.2}$$

for all $x \in I$. Denote by $\mathrm{AC}(I)$ the set of all absolutely continuous functions on an interval I.

Theorem: *Let $r \geq 1$ be a real number and let a positive continuous function ρ satisfy* (5.2). *Let us set*

$$W(t) \equiv \int_a^t \rho(\xi)\,d\xi \text{ and } K_\rho(s,t) \equiv \int_a^{s\wedge t} \rho(v)\,dv = W(\,s\wedge t\,) \quad (s,t \in I), \tag{5.3}$$

where $s \wedge t \equiv \min(s,t)$. The reproducing kernel Hilbert space $H_{(K_\rho)^r}(I)$ and its norm are given by:

$$H_{(K_\rho)^r}(I) \equiv \left\{ f \in \mathrm{AC}(I) \,:\, f(a)=0,\, f' \in L^2(I, W^{1-r}\rho^{-1}\,dt) \right\}, \tag{5.4}$$

and

$$\|f\|_{H_{(K_\rho)^r}(I)} \equiv \left(\frac{1}{r} \int_I |f'(t)|^2 W(t)^{1-r} \rho(t)^{-1}\,dt \right)^{\frac{1}{2}},$$

respectively.

(For [14], page 17, in (1.59) put the factor $\frac{1}{r}$.)

For any positive integers m,n

$$K_\rho(s,t)^{m+n} = K_\rho(s,t)^m K_\rho(s,t)^n,$$

and so we obtain the corresponding norm inequality

$$\int_a^b |(f_1(x)f_2(x))'|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{m+n-1} \rho(x)}$$

$$\leq \left(\frac{1}{m} + \frac{1}{n} \right) \int_a^b |f_1'(x)|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{m-1} \rho(x)} \int_a^b |f_2'(x)|^2 \frac{dx}{\left(\int_a^x \rho(t)dt\right)^{n-1} \rho(x)}.$$

We note that

Open problem: *How will be the inequality for noninteger case m,n?*

Meanwhile, from the identity

$$K_{H(a,b;\mathbf{R})}(x,x_1) K_{H(a',b';\mathbf{R})}(x,x_1) = \frac{1}{2}\left(\frac{a}{b} + \frac{a'}{b'} \right) K_{H(aa',ab'+a'b;\mathbf{R})}(x,x_1),$$

we obtain the corresponding inequality

$$\int_{\mathbf{R}} \left((aa')^2|(f(x)g(x))'|^2 + (ab' + a'b)^2|f(x)g(x)|^2\right) dx$$
$$\leq \frac{1}{2}\left(\frac{a}{b} + \frac{a'}{b'}\right) \int_{\mathbf{R}} \left(a^2|f'(x)|^2 + b^2|f(x)|^2\right) dx \int_{\mathbf{R}} \left((a')^2|g'(x)|^2 + (b')^2|g(x)|^2\right) dx.$$

Infinite Order Sobolev Spaces

Note that the kernel

$$K(z, \overline{u}; t) = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{1}{8t}(z - \overline{u})^2\right)$$

is the reproducing kernel for the Hilbert space with finite norms

$$\sqrt{\sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 dx} = \sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy}. \quad (5.5)$$

These mean that for the restriction to the real line, the Hilbert space is an infinite order Sobolev Hilbert space and on the complex plane the space is composed of entire functions ([14], pages 141-145). We thus have the identity

$$K(z, \overline{u}; t/2) = 4\sqrt{\pi t} K(z, \overline{u}; t)^2$$

and we have the corresponding norm inequality.

For the isometric inequality (5.1) for the Bergman and Szegö spaces, note their representations ([14], pages 146-147).

We write $S(r) \equiv \{z \in \mathbb{C} : 0 < \arg(z) < r\}$ for the open sector and its boundary $\partial S(r) \equiv \{z \in \mathbb{C} : z = 0 \text{ or } \arg(z) = \pm r\}$.

(Note that we defined as

$$\arg 0 = 0$$

as a result of the division by zero in [15].)

Theorem H: *Let $r \in (0, \pi/2)$. For an analytic function f on the open sector $S(r)$, we have the identity*

$$\iint_{S(r)} |f(x+iy)|^2 dxdy = \sin(2r) \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j+1)!} \int_{\mathbb{R}} x^{2j+1} |f^{(j)}(x)|^2 dx. \quad (5.6)$$

Conversely, if any $f \in C^{\infty}(\partial S(r))$ has a convergent sum in the right-hand side in (5.6)*, then the function $f(x)$ can be extended analytically onto the sector $S(r)$ in the form $f(z)$ and the identity* (5.6) *is valid.*

In the Szegö space, we have the following formula:

Theorem I: *Let $r \in (0, \pi/2)$. For any member f in the Szegö space on the open sector $S(r)$, we have the identity*

$$\oint_{\partial S(r)} |f(z)|^2 |dz| = 2\cos r \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j)!} \int_{\mathbb{R}} x^{2j} |f^{(j)}(x)|^2 dx, \tag{5.7}$$

where $f(x)$ means the nontangential Fatou limit on $\partial S(r)$ for $x \in \mathbb{R}$. Conversely, if any $f \in C^{\infty}(0, \infty)$ has a convergent sum in the right-hand side in (5.7)*, then the function $f(x)$ extends analytically onto the open sector $S(r)$ and the identity* (5.7) *is valid.*

5.3. Fischer Spaces. As a simple case, we shall refer to the Fischer space $\mathcal{F}_a(\mathbb{C})$ admitting the reproducing kernel, for any fixed $a > 0$

$$K_a(z, \overline{u}) = \exp(a^2 z\overline{u}) \quad (z, u \in \mathbb{C})$$

with finite norms for entire functions $f(z)$

$$\|f\|_{\mathcal{F}_a(\mathbb{C})} = a\sqrt{\frac{1}{\pi} \iint_{\mathbb{C}} |f(z)|^2 \exp(-a^2|z|^2)\, dx\, dy}.$$

See [14], page 170. We thus have the relation for any positive a, b

$$K_a(z, \overline{u}) K_b(z, \overline{u}) = K_{\sqrt{a^2+b^2}}(z, \overline{u})$$

and the corresponding result.

An experiment:

For the case $a = 1$, and for $K_1 = K_1(z, \overline{u})$ and $K = K_1(z, \overline{u})$ and so, we have $K_1 K = K_{\sqrt{2}}(z, \overline{u})$. Then, for $g = f_1(z)e^z = 1 \in H_{K_1 K_1} = H_{K_{\sqrt{2}}} = H_{\sqrt{2}}$, we obtain the curious result

$$f_1(z) = e^{z/2+1/2} = \left(e^{z_2} \left(1, e^{z_1} K_1(z_1, \overline{z_2})\right)_{H_{\sqrt{2}}}, e^{z_2} K_1(z_2, \overline{z})\right)_{H_{\sqrt{2}}}, \tag{5.8}$$

when assuming the formula (3.10). This means that the identity assumption is not valid.

In order to see the formula (3.8), we calculate

$$L^* L K(z, \overline{u}) = L^* L e^{z\overline{u}} = \exp\left(\frac{1}{2}\right) \exp\left(\frac{z}{2}\right) \exp\left(\frac{\overline{u}}{2}\right) \exp\left(\frac{1}{2} z\overline{u}\right)$$

and

$$L^* L L^* L K(z, \overline{u}) = \exp\left(\frac{5}{4}\right) \exp\left(\frac{3z}{4}\right) \exp\left(\frac{3\overline{u}}{4}\right) \exp\left(\frac{1}{4} z\overline{u}\right).$$

We can realize the spaces H_{KK} and H_k, concretely and analytically, however, the application of the formula (3.8) is involved analytically.

Indeed, the reproducing kernel Hilbert space H_k is realized as follows:

Any member $f(z)$ of H_k is represented with the form

$$f(z) = C_* \exp\left(\frac{5}{4}\right) \exp\left(\frac{3z}{4}\right) \sum_{n=0}^{\infty} \frac{C_n z^n}{4^n n!}$$

with a finite norm

$$||f||_{H_k}^2 = |C_*|^2 \exp\left(\frac{5}{4}\right) \sum_{n=0}^{\infty} \frac{|C_n|^2}{4^n n!},$$

as we see from the expansion of the kernel k. Then, the corresponding coefficients are given as follows:

For $L^* LK(z, \overline{u})$

$$C_* = \exp\left(\frac{-3}{4}\right)\left(\frac{\overline{u}}{2}\right), \quad C_n = (2\overline{u} - 1)^n.$$

For $L^* 1$,

$$C_* = \exp\left(\frac{-5}{4}\right), \quad C_n = (-3)^n.$$

By using these results, by the formula (3.8), we have the right result

$$f_{\mathbf{d}} = e^{-z},$$

that is, it is the natural solution to the identity

$$f_{\mathbf{d}} e^z = 1.$$

5.4. Matrices case. Meanwhile, any positive definite Hermitian matrix may be considered as a reproducing kernel and so we can apply the theory of reproducing kernels to that of positive definite Hermitian matrices ([7, 9]). For the product of two positive definite Hermitian matrices A, B with the same size, and for the Hadamard product $*$ and for the complex conjugate transpose $*$, we can state the results as in

$$\left(\mathbf{x}^{(1)} * \mathbf{x}^{(2)}\right)^* \left(A^{-1} * B^{-1}\right)^{-1} \left(\mathbf{x}^{(1)} * \mathbf{x}^{(2)}\right) \leq \left(\mathbf{x}^{(1)\ *} A \mathbf{x}^{(1)}\right) \left(\mathbf{x}^{(2)\ *} B \mathbf{x}^{(2)}\right)$$

and

$$\left(A^{-1} * B^{-1}\right)^{-1} \leq A * B,$$

symbolically ([7], page 128). Equality problems are all solved .

6. Division by Zero Calculus

We note that the famous division by zero $1/0$, $0/0$ and any fractional $g/0$ for $f \equiv 0$ are 0 and the zero function, respectively, trivially in our sense in the both senses of any Tikhonov functionals with any parameter $\alpha > 0$ and the Moore-Penrose generalized solutions $\alpha = 0$. See [15, 16] for the details.

If $b = 0$ in $K_{H(a,b;\mathbf{R})}(x, x_1)$, then, by the division by zero calculus

$$K_{H(a,0;\mathbf{R})}(x, x_1) = -\frac{1}{2a^2}|x - x_1|$$

and this is the reproducing kernel for the corresponding space $H(a, 0; \mathbf{R})$ equipped with the norm

$$\|f\|_{H(a,0;\mathbf{R})}^2 = a^2 \int_0^a (f'(x)^2 dx.$$

See [15, 16] for the division by zero calculus. Note that it is the Green's function in one dimensional space on the whole space and the Green's function may be related to the reproducing kernel. See [14], pages 62-63.

Meanwhile, if $a = 0, K_{H(0,b;\mathbf{R})}(x, x_1) = 0$, then it is the trivial reproducing kernel for the zero function space.

However, from the representation

$$\frac{1}{2ab} \exp\left(-\frac{b}{a}|x-y|\right) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \frac{e^{i\xi(x-y)} d\xi}{a^2\xi^2 + b^2},$$

for $a = 0$, we have the reasonable result

$$\frac{1}{b^2}\delta(x-y)$$

that may be considered as the reproducing kernel for the L_2 space. See Section 8.8 in [14].

Acknowledgments.

The author wishes to express his deep thanks Professor Akira Yamada for his great contributions to the paper. Professors Yoshihiro Sawano and Michio Seto invited Professor Akira Yamada and the author to the research meeting:

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at the RIMS, Kyoto University and so the author would like to express his deep thanks to them for their great contribution with the RIMS. Indeed, this topic was considered for the research meeting and in the meeting, some part was presented.

This research and the publication cost were supported by
KAKENHI-PROJECT-20K03728/

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