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WHAT IS A REPRODUCING KERNEL? - SOME ESSENCES FOR THE NEW JOURNAL

TSUTOMU MATSUURA AND SABUROU SAITOH

Abstract. The theory of reproducing kernels is very fundamental, beautiful and has many applications in analysis and numerical analysis as in the Pythagorean theorem. As a paper in the first volume of the new journal, we would like to refer to some general viewpoint for reproducing kernels and some essences of reproducing kernels based on the authors' viewpoint. However, the theory is already much more developing widely and deeply on this century.

The contents are as follows:

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1. INTRODUCTION AND SOME GLOBAL VIEWPOINT ON REPRODUCING KERNELS

Since S. Zaremba (1907) and S. Bergman (1922), around one century before, the reproducing kernel theory was expanded very widely and deeply through N. Aronszajn (1950) as in

1. One Complex Variable CV
2. Several Complex Variable SCV
3. Several Real Variables and Harmonic Functions SRVH
4. Abstract Theory and Operator Theory ATOT
5. Integral Transforms and Integral Equations ITIE
6. Kernel Methods KM
7. Probability and Statistics Theory PST
8. Numerical Analysis NA

and others

9. Others OT.

From the viewpoint of some general interest, we published the book [62] which does not contain great results in one and several complex analysis and in the last part, we referred to as in the followings:

For the theory of reproducing kernels in one-variable complex analysis, see the fundamental book [8] by S. Bergman. Indeed, for some long years, we considered the Bergman kernel as the theory of reproducing kernels and related published papers were vast. They called **kernel functions** for reproducing kernels. For the advanced and profound theory, see the books D. A. Hejhal [36] and J. D. Fay [24] in connection with the Riemann theta functions and the Klein prime form. For their applications, see A. Yamada [70]. Their theory now seems to be, however, too deep to deal with for any mathematician. For the theory of reproducing kernels in several-variable complex analysis, see the classical books [30, 31]. However, the theory in several complex analysis is developing greatly and its situation may be looked in the paper [5] by E. Barletta, S. Dragomir and F. Esposito in this volume. For the old history of reproducing kernels, see N. Aronszajn [4, 54]. In the Proceedings [56] of an international conference, we can find various results on the theory of reproducing kernels. See D. Alpay [2] for wider subjects on reproducing kernels.

We find many results in the **learning theory** where the applications of the theory of reproducing kernels are important; that is, the estimation of covering numbers by the disks of reproducing kernel Hilbert spaces as subspaces of a family of continuous functions, some detailed smoothness relationships between reproducing kernels and reproducing kernel Hilbert spaces, and approximations of functions by Sobolev spaces. See, for example, F. Cucker and S. Smale [20] and D. X. Zhou [72]. Indeed, we have many references for the learning theory, see [68].

For **Support Vector Machines** that are favorable for engineers, see [19] and its research center is MIT.

We would like to note the active research results of K. Fukimizu [21, 32, 33] on the statistic theory and reproducing kernels.

The article D. Alpay [1] summarizes various applications of the theory in connection with operator theory on Hilbert spaces.

For the connection with the stochastic theory and reproducing kernels, see Berlinet [10, 11] who was very active in the 5th ISAAC Catania Congress with his colleagues and we see many results in this field.

For some more recent general discretization principle with many concrete applications, see [15, 16]. A new global theory combining the fundamental relations among eigenfunctions, initial value problems in general linear partial differential operators, and reproducing kernels, see [17].

In this paper, some general essences from the viewpoint of general applications and general interest will be introduced by our results. In particular, on this line, reproducing kernels in complex analysis that are typically stated as the Bergman kernels in one and several complex variables will not be referred to here. For the vast world of reproducing kernels, we would like to refer to some essences of reproducing kernels from the viewpoint of the authors. Indeed, the authors wanted to show some core of reproducing kernels for their vast world. We wanted to be this paper in a self contained manner, because modern mathematics is advanced and will not be understood for many mathematical scientists.

2. WHAT IS A REPRODUCING KERNEL?

First of all, we would like to state that *what is a reproducing kernel?*

In general, a complex-valued function $k : E \times E \to \mathbb{C}$ is called a *positive definite quadratic form function* on an abstract set E, or shortly, *positive definite function*, when it satisfies the property that, for an arbitrary function $X : E \to \mathbf{C}$ and for any finite subset F of E,

$$\sum_{p,q \in F} X(p)\overline{X(q)}k(p,q) \geq 0.$$

Therefore, the simplest positive definite quadratic form functions are

$$f(p)\overline{f(q)}$$

for any function $f(p)$ and for any set E.

Then we obtain the fundamental result:

Proposition 2.1. *For any positive definite quadratic form function $k : E \times E \to \mathbb{C}$, there exists a uniquely determined reproducing kernel Hilbert space $H_k = H_k(E)$ admitting the reproducing kernel k on E whose characterization is given by the* *two properties:* (i) $k(\cdot,q) \in H_k$ *for any* $q \in E$ *and,* (ii) *for any* $f \in H_k$ *and for any* $p \in E$, $(f(\cdot), k(\cdot.p))_{H_k} = f(p)$.

The properties (i) and (ii) are called the reproducing property of the function $k(p,q)$ in the Hilbert space $H_k = H_k(E)$.

For the realization of the space $H_k = H_k(E)$ in terms of k we have many methods, however, the Aveiro discritzation method will give the simplest and general method as in stated in the later. The idea comes from the general and simple discretization and from improving the power of computers.

Apparently, any reproducing kernel is a positive definite quadratic form function and the one to one correspondence is very fundamental and important.

3. WHAT IS THE THEORY OF REPRODUCING KERNELS ?

The important correspondence between a positive definite quadratic form function (: reproducing kernel, or kernel, RK) and the associated reproducing kernel Hilbert space (: RKHS) is proposing various problems between the two concepts. For positive definite quadratic form functions, for example, sum and product are again positive definite quadratic form functions and so, we can consider the related reproducing kernel Hilbert spaces. In this direction we can consider many and many problems like

restriction of kernels,
pullback of a reproducing kernel,
pasting of reproducing kernels,
and so on.

Meanwhile, conversely, from the structure of reproducing kernel Hilbert spaces, we can consider the associated reproducing kernels like

tensor products of reproducing kernel Hilbert spaces,
balloon of a reproducing kernel Hilbert space,
increasing sequence of reproducing kernel Hilbert spaces,
wedge product of reproducing kernel Hilbert spaces,
and so on.

Of course, the corresponding properties like smoothness between reproducing kernels and the associated reproducing kernel functions are fundamental problems.

In those senses, we can say that the theory of reproducing kernels are to examine the relations of positive definite functions and the associated reproducing kernel Hilbert spaces:

RKs $\Longleftrightarrow$ RKHSs.

The typical and fundamental results are the restriction, sum and product properties as in the followings.

Now suppose that we are given a positive definite quadratic form function $K : E \times E \to \mathbb{C}$. We shall consider restriction of K to $E_0 \times E_0$, where E_0 is a subset of E. Of course, the restriction is again a positive definite quadratic form function on the subset $E_0 \times E_0$. We can consider the relation between the two reproducing kernel Hilbert spaces.

Proposition 3.1. *Suppose that $K : E \times E \to \mathbb{C}$ is a positive definite quadratic form function on a set E. Let E_0 be a subset of E. Then the Hilbert space that $K|E_0 \times E_0 : E_0 \times E_0 \to \mathbb{C}$ defines is given by:*

$$H_{K|E_0\times E_0}(E_0) = \{f \in \mathcal{F}(E_0) \,:\, f = \tilde{f}|E_0 \text{ for some } \tilde{f} \in H_K(E)\}. \tag{3.1}$$

Furthermore, the norm is given in terms of the one of $H_K(E)$:

$$\|f\|_{H_{K|E_0\times E_0}(E_0)} = \min\{\|\tilde{f}\|_{H_K(E)} \,:\, \tilde{f} \in H_K(E),\, f = \tilde{f}|E_0\}. \tag{3.2}$$

Suppose that we are given two positive definite quadratic functions $K_1, K_2 : E \times E \to \mathbb{C}$. The usual sum $K(p,q) = K_1(p,q) + K_2(p,q)$ on $E \times E$ is also a positive definite quadratic function on E. We consider the relation between the corresponding reproducing kernel Hilbert spaces $H_{K_1}(E)$, $H_{K_2}(E)$ and $H_K(E)$.

Proposition 3.2. *Let $K_1, K_2 : E\times E \to \mathbb{C}$ be positive definite. Set $K \equiv K_1 + K_2$.*

(1) *We have*

$$H_K(E) = \bigcup\{f_1 + f_2 \in \mathcal{F}(E) \,:\, f_1 \in H_{K_1}(E), f_2 \in H_{K_2}(E)\},$$

and as a linear space, we have $H_{K_1+K_2}(E) = H_{K_1}(E) + H_{K_2}(E)$.

(2) *The norm of $H_K(E)$ is given in terms of those of $H_{K_1}(E)$ and $H_{K_2}(E)$:*

$$\|f\|_{H_K} = \min_{\substack{f_1\in H_{K_1}(E),\, f_2\in H_{K_2}(E),\\ f=f_1+f_2}} \sqrt{\|f_1\|_{H_{K_1}(E)}{}^2 + \|f_2\|_{H_{K_2}(E)}{}^2}. \tag{3.3}$$

Given two Hilbert spaces H_1 and H_2, we consider the tensor product $H_1 \otimes H_2$ with the inner product:

$$\langle h_1 \otimes h_2, h_1' \otimes h_2'\rangle_{H_1\otimes H_2} = \langle h_1, h_1'\rangle_{H_1}\langle h_2, h_2'\rangle_{H_2} \quad (h_1, h_1' \in H_1, h_2, h_2' \in H_2).$$

When we are given two complex-valued functions f and g defined on E_1 and E_2, the operation is available: $(p_1,p_2) \in E_1 \times E_2 \to f \otimes g(p_1,p_2) \equiv f(p_1)g(p_2) \in \mathbb{C}$. Now we consider the product of positive definite quadratic functions $K_1 : E_1 \times E_1 \to \mathbb{C}$ and $K_2 : E_2 \times E_2 \to \mathbb{C}$. Then, we have

Proposition 3.3. *Let $K_1 : E_1 \times E_1 \to \mathbb{C}$ and $K_2 : E_2 \times E_2 \to \mathbb{C}$ be positive definite quadratic functions. Then $K_1 \otimes K_2 : E_1 \times E_2 \times E_1 \times E_2 \to \mathbb{C}$ is a positive definite quadratic function and*

$$H_{K_1}(E_1) \otimes H_{K_2}(E_2) = H_{K_1\otimes K_2}(E_1 \times E_2). \tag{3.4}$$

In particular, the case $E_1 = E_2 = E$ is important. Now we consider its restriction to the diagonal set. Let us set

$$D = \{(p,q,p,q) \,:\, p,q \in E\}. \tag{3.5}$$

Then we see that $K_1 \otimes K_2|D$ is a positive definite quadratic function. An immediate consequence is the following:

Proposition 3.4. *Suppose that $K_1, K_2 : E\times E \to \mathbb{C}$ are positive definite quadratic functions. Then so is the pointwise product $K \equiv K_1 \cdot K_2 : E \times E \to \mathbb{C}$.*

Proposition 3.5. *Let $K_1, K_2 : E \times E \to \mathbb{C}$ be positive definite quadratic functions. Then, we obtain*

$$\|f_1 + f_2\|_{H_{K_1+K_2}(E)} \leq \|f_1\|_{H_{K_1}(E)} + \|f_2\|_{H_{K_2}(E)} \tag{3.6}$$

and

$$\|f_1 \cdot f_2\|_{H_{K_1 K_2}(E)} \leq \|f_1\|_{H_{K_1}(E)} \|f_2\|_{H_{K_2}(E)} \tag{3.7}$$

for $f_1 \in H_{K_1}(E)$ and $f_2 \in H_{K_2}(E)$.

Proposition 3.6. *Let $K : E \times E \to \mathbb{C}$ be a positive definite quadratic function and let s be a mapping from E to non-vanishing $\mathbb{C}$. Define $K_s(p,q) \equiv s(p)\overline{s(q)}K(p,q)$ for $p, q \in E$. Then we have*

$$H_{K_s}(E) = \{F \in \mathcal{F}(E) \,:\, F = f \cdot s \text{ for some } f \in H_K(E)\}. \tag{3.8}$$

Furthermore, one has $\langle f \cdot s, g \cdot s\rangle_{H_{K_s}(E)} = \langle f, g\rangle_{H_K(E)}$ for all $f, g \in H_K(E)$.

A trivial inquality (3.7) is a very strong technique which produces many inequalities. Furthermore, we can realize many reproducing kernel Hilbert spaces from known reproducing kernel Hilbert spaces by using Proposition 3.6 and therefore, the technique is a very important with the related norm inequalities. See, for example, [55].

Meanwhile, the multiply product of a reproducing kernel will give the basic relation of the related linear mapping and nonlinear mappings, as we shall see later. See also, for example, [55].

The differential properties of functions and kernels are also important and beautiful.

Define

$$C^{1,1}(O \times O) \equiv \{f \in C(O \times O) \,:\, \partial_x f, \partial_y f,\, \partial_x\partial_y f, \partial_y\partial_x f \text{ exists and continuous in } O\}.$$

In terms of $C^{1,1}(O \times O)$, we have the following result

Proposition 3.7. *Let O be an open set in $\mathbb{R}$. If a positive definite function $K \in C^{1,1}(O \times O)$, then $\partial x \partial y K$ is also a positive definite kernel and we have*

$$\|f'\|_{H_{\partial x \partial y K(O)}} \leq \|f\|_{H_K(O)}. \tag{3.9}$$

If $H_K(O)$ contains 0 as the unique constant function, then we have

$$\|f'\|_{H_{\partial x \partial y K(O)}} = \|f\|_{H_K(O)}. \tag{3.10}$$

From this viewpoint: **RKs** $\Longleftrightarrow$ **RKHSs**, we think that the problems are expanding endlessly and deeply. In general, inverse or converse is difficult, at this moment, we can say that the direction **RKs** $\Longleftarrow$ **RKHSs** is weak.

4. WHY REPRODUCING KERNELS ARE FUNDAMENTAL AND IMPORTANT ?

Now, let $\mathcal{H}$ be a Hilbert (possibly finite-dimensional) space, and consider E to be an abstract set and $\mathbf{h}$ a Hilbert $\mathcal{H}$-valued function on E. Then, a very general linear transform from $\mathcal{H}$ into the linear space $\mathcal{F}(E)$ comprising all the complex valued functions on E may be considered by

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H}, \tag{4.1}$$

in the framework of Hilbert spaces. Many general linear mappings may be considered in this framework by many modifications and arrangements. For this recall the Schwartz kernel theorem in connection with the distribution theory. In particular, recall that distribution may be considered as functions by considering integrals. Further, restrictions of functions and weighted norms are usual techniques. However, the theory of reproducing kernels may be considered essentially and favorably in the framework of Hilbert spaces. However, even for the theory of Banach spaces, we can discuss the relation between the spaces of Banach and Hilbert, and we can find many related references.

In order to investigate the linear mapping (4.1), we form a positive definite quadratic form function $K(p, q)$ on $E \times E$ defined by

$$K(p, q) = (\mathbf{h}(q), \mathbf{h}(p))_{\mathcal{H}} \quad \text{on} \quad E\times E. \tag{4.2}$$

Then, the following fundamental results are valid:

Proposition 4.1. (I) *The range of the linear mapping* (4.1) *by* $\mathcal{H}$ *is characterized as the reproducing kernel Hilbert space* $H_K(E)$ *admitting the reproducing kernel* $K(p, q)$*.*

(II) *In general, the inequality*

$$\|f\|_{H_K(E)} \leq \|\mathbf{f}\|_{\mathcal{H}}$$

holds. Here, for any member f *of* $H_K(E)$ *there exists a uniquely determined* $\mathbf{f}^* \in \mathcal{H}$ *satisfying*

$$f(p) = (\mathbf{f}^*, \mathbf{h}(p))_{\mathcal{H}} \quad \text{on } E$$

and

$$\|f\|_{H_K(E)} = \|\mathbf{f}^*\|_{\mathcal{H}}. \tag{4.3}$$

(III) *In general, the inversion formula in* (4.1) *in the form*

$$f \mapsto \mathbf{f}^* \tag{4.4}$$

in (II) *holds, by using the reproducing kernel Hilbert space* $H_K(E)$*.*

When we consider the linear mapping (4.1), Proposition 4.1 gives the image identification that is a very fundamental and important. Furthermore, we can obtain many and many reproducing kernels from the general linear mappings. When we consider the inversion of the linear mapping (4.1), the typical ill-posed problem (4.1) looking for its inversion becomes a well-posed problem, because the image space of (4.1) is characterized as the reproducing kernel Hilbert space $H_K(E)$ with the isometric identity (4.3), which may be considered as a generalization of the *Pythagorean* theorem.

However, this viewpoint is a mathematical one and is not a numerical one and not easy to deal with analytical and numerical problems.

In Proposition 4.1, when we know the isometrical mapping between the domain $\mathcal{H}$ and the reproducing kernel Hilbert space $H_K(E)$, we can determine the system or linear mapping (4.1) by representing the system function $\mathbf{h}$ in terms of the linear mapping and the reproducing kernel. This important concept and applications seem to be still weak for the image identification and inversion problems.

Here, we shall introduce the typical image identifications and isometric identities that are obtained by considering the integral representations in the heat conduction.

We consider the simple heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbb{R} \times T_+ \quad (T_+ \equiv \{t > 0\}) \tag{4.5}$$

subject to the initial condition

$$u_F(\cdot, 0) = F \in L^2(\mathbb{R}) \quad \text{on} \quad \mathbb{R}. \tag{4.6}$$

Using the Fourier transform, we obtain a representation of the solution $u_F(x,t)$

$$u_F(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} F(\xi) \exp\left(-\frac{(x-\xi)^2}{4t}\right) d\xi \tag{4.7}$$

at least in the formal sense.

For any fixed $t > 0$, we first examine the integral transform $F \mapsto u_F$ and we shall characterize the image function $u_F(x,t)$.

We write

$$k(x;t) \equiv \frac{1}{\sqrt{4\pi t}} \exp\left(-\frac{x^2}{4t}\right) \quad (x \in \mathbb{R},\, t > 0) \tag{4.8}$$

and

$$K(x,x';t) \equiv \int_{\mathbb{R}} k(x-\xi;t)k(x'-\xi;t)d\xi = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{x^2}{8t} - \frac{x'^2}{8t} + \frac{xx'}{4t}\right). \tag{4.9}$$

Then, we obtained surprisingly the initial results.

Proposition 4.2. *Let $t > 0$ fix. A function f takes the form $u_F(\cdot,t)$ for some $F \in L^2(\mathbb{R})$ if and only if f admits analytic extension $\tilde{f}$ to $\mathbb{C}$ and satisfies*

$$\sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(\frac{-y^2}{2t}\right) dxdy} < \infty. \tag{4.10}$$

In this case, $f \in H_K(\mathbb{R})$ and the norm is given by:

$$\|f\|_{H_K(\mathbb{R})} = \sqrt{\frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |\tilde{f}(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy}.$$

Proposition 4.3. *Let $t > 0$ fix. In the integral transform $F \mapsto u_F(\cdot, t)$ of $L^2(\mathbb{R})$ functions F, the images $u_F(\cdot, t)$ extend analytically onto $\mathbb{C}$ to a function, which we still write $u_F(\cdot, t)$. Furthermore, we have the isometrical identity*

$$\int_{\mathbb{R}} |F(\xi)|^2 d\xi = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |u_F(z,t)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy, \tag{4.11}$$

for any fixed $t > 0$.

Proposition 4.4. *If a C^∞-function $f : \mathbb{R}^n \to \mathbb{C}$ has a finite integral on the right-hand side in (4.11), then f is extended analytically onto $\mathbb{C}$ and*

$$\sum_{j=0}^{\infty} \frac{(2t)^j}{j!} \int_{\mathbb{R}} |\partial_x^j f(x)|^2 dx = \frac{1}{\sqrt{2\pi t}} \iint_{\mathbb{C}} |f(x+iy)|^2 \exp\left(-\frac{y^2}{2t}\right) dxdy. \tag{4.12}$$

Proof. For fixed $t > 0$, the solution operator $F \mapsto u_F$ can be regarded as

$$u_F(x,t) = \langle F, k(x - \cdot; t)\rangle_{L^2(\mathbb{R})}.$$

So with

$$E = \mathbb{R}, \quad \mathcal{H} = L^2(\mathbb{R}), \quad \mathbf{h}(x) \equiv k(x - \cdot; t) \text{ and } L = [F \in \mathcal{H} \mapsto u_F(\cdot, t) \in \mathcal{F}(E)],$$

the reproducing Hilbert kernel space $\mathcal{R}(L)$ is given by:

$$\mathcal{R}(L) = \{u_F(\cdot, t) \,:\, F \in L^2(\mathbb{R})\}.$$

Note that for any fixed $t > 0$, the system

$$\{k(\cdot - \xi, t); \xi \in \mathbb{R}\} \tag{4.13}$$

spans a dense subspace in $L^2(\mathbb{R})$. Therefore, L is isometric.

Now let us view this mapping from the point of complex analysis. Note that the kernel $K(x, x'; t)$ extends analytically to $\mathbb{C} \times \overline{\mathbb{C}}$;

$$K(z, \overline{u}; t) = \frac{1}{2\sqrt{2\pi t}} \exp\left(-\frac{z^2}{8t} - \frac{\overline{u}^2}{8t} + \frac{z\overline{u}}{4t}\right). \tag{4.14}$$

Observe that (4.14) stands for

$$K(z, \overline{u}; t) = \int_{\mathbb{R}} k(z - \xi; t)\overline{k(u - \xi; t)}\, d\xi. \tag{4.15}$$

Consequently, the extended kernel K is again a positive definite function. Denote by $H_K(\mathbb{C})$ the RKHS associated with K. The following is a description of $H_K(\mathbb{C})$; we can see directly

$$H_K(\mathbb{C}) = \left\{ f \in \mathcal{O}(\mathbb{C}) \,:\, \|f\|_{H_K(\mathbb{C})} = \sqrt{\iint_{\mathbb{C}} \frac{|\tilde{f}(x+iy)|^2}{\sqrt{2\pi t}} \exp\left(\frac{-y^2}{2t}\right) dxdy < \infty} \right\}. \tag{4.16}$$

However, (4.16) was derived naturally and simply from the reproducing structure of (4.14) by using Proposition 3.6.

Meanwhile, the norm (4.10) is also expressible in terms of the trace $f(x)$ of $\tilde{f}(z)$ to the real line.

By using the identity

$$k(x-\xi,t)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp\{-p^2t+ip(x-\xi)\}dp, \tag{4.17}$$

we have

$$K(x,x';t)=\frac{1}{2\pi}\int_{\mathbb{R}}\exp\{-2p^2t+ip(x-x')\}dp. \tag{4.18}$$

This implies that any member $f(x)$ of $H_K(\mathbb{R})$ is expressible in the form

$$f(x)=\frac{1}{2\pi}\int_{\mathbb{R}}g(p)\exp(ipx-2p^2t)dp=\frac{1}{\sqrt{2\pi}}\mathcal{F}^{-1}[g\cdot\exp(-2p^2t)](x) \tag{4.19}$$

for a function g satisfying

$$\int_{\mathbb{R}}|g(p)|^2\exp(-2p^2t)dp<\infty \tag{4.20}$$

and we have the isometrical identity

$$\|f\|_{H_K(\mathbb{R})}=\sqrt{\frac{1}{2\pi}\int_{\mathbb{R}}|g(p)|^2\exp(-2p^2t)dp}. \tag{4.21}$$

Meanwhile, by the Fourier transform and (4.19), we have

$$g(p)=\sqrt{2\pi}\mathcal{F}f(p)\exp(2p^2t) \tag{4.22}$$

in $L^2(\mathbb{R})$. Hence, we obtain

$$\|f\|_{H_K(\mathbb{R})}{}^2=\int_{\mathbb{R}}|\mathcal{F}f(p)|^2\exp(-2p^2t)dp.=\sum_{j=0}^{\infty}\frac{(2t)^j}{j!}\int_{\mathbb{R}}|f^{(j)}(x)|^2dx, \tag{4.23}$$

by virtue of the monotone convergence theorem and the Parseval–Plancherel identity.

□

As in those typical cases, we obtained many isometric identities and analytic extension formulas. Define the right-half plane by: $R^+=\{z\in\mathbb{C}:\mathrm{Re}(z)>0\}$.

Proposition 4.5. *Let $q>\frac{1}{2}$. Then, for $f\in H_{K_q}(R^+)$, admitting the Bergman Selberg reproducing kernel $\Gamma(2q)/(z+\overline{u})^{2q}$, we have the identity*

$$\|f\|_{H_{K_q}(R^+)}\equiv\sqrt{\frac{1}{\Gamma(2q-1)\pi}\iint_{R^+}|f(z)|^2(2x)^{2q-2}dxdy}$$
$$=\sqrt{\sum_{n=0}^{\infty}\frac{1}{n!\Gamma(n+2q+1)}\int_{\mathbb{R}}|(xf'(x))^{(n)}|^2x^{2n+2q-1}dx}. \tag{4.24}$$

Conversely, any $C^\infty(0,\infty)$-function f with convergent summation in (4.24) extends analytically onto the right half plane R^+. The analytic extension $f(z)$ satisfying $\lim_{x\to\infty}f(x)=0$ belongs to $H_{K_q}(R^+)$ and the identity (4.24) is valid.

We shall write $S(r) \equiv \{z \in \mathbb{C} : 0 < \arg(z) < r\}$ for the open sector and its boundary $\partial S(r) \equiv \{z \in \mathbb{C} : z = 0 \text{ or } \arg(z) = \pm r\}$.

Proposition 4.6. *Let $r \in (0, \pi/2)$. For an analytic function f on the open sector $S(r)$, we have the identity*

$$\iint_{S(r)} |f(x+iy)|^2 dxdy = \sin(2r) \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j+1)!} \int_{\mathbb{R}} x^{2j+1} |f^{(j)}(x)|^2 dx. \tag{4.25}$$

Conversely, if any $f \in C^\infty(S(r))$ has a convergent sum in the right-hand side in (4.25)*, then the function $f(x)$ can be extended analytically onto the sector $S(r)$ in the form $f(z)$ and the identity* (4.25) *is valid.*

In the Szegö space, we have the following formula:

Proposition 4.7. *Let $r \in (0, \pi/2)$. For any member f in the Szegö space on the open sector $S(r)$, we have the identity*

$$\oint_{\partial S(r)} |f(z)|^2 |dz| = 2\cos r \sum_{j=0}^{\infty} \frac{(2\sin r)^{2j}}{(2j)!} \int_{\mathbb{R}} x^{2j} |f^{(j)}(x)|^2 dx, \tag{4.26}$$

where $f(x)$ means the nontangential Fatou limit on $\partial S(r)$ for $x \in \mathbb{R}$. Conversely, if any $f \in C^\infty(0, \infty)$ has a convergent sum in the right-hand side in (4.26)*, then the function $f(x)$ extends analytically onto the open sector $\Delta(r)$ and the identity* (4.26) *is valid.*

These results were applied to investigate analyticity properties of the solutions of nonlinear partial differential equations. In particular, an analogue is applied to the proof of the unique existence of the Schödinger equation by N. Hayashi. H. Aikawa considered the class $W(c_j; \mathbb{R})$ by changing $\frac{(2\sin r)^{2j}}{(2j)!}$ with other general positive sequences, where he proved that some function can not be extended beyond a sector.

These Propositions may be looked the relations of the correspondent RKHSs for the restrictions of RKs, because the both sides are reproducing kernel Hilbert spaces. That is the relation of analytic extension and the restriction (extension) of reproducing kernel.

5. GENERALIZED REPRODUCING KERNELS AND GENERALIZED DELTA FUNCTIONS

We shall consider a family of *any complex valued functions* $\{U_n(p)\}_{n=0}^{\infty}$ defined on an abstract set E that are linearly independent. Then, we consider the form:

$$K_N(p,q) = \sum_{n=0}^{N} U_n(p)\overline{U_n(q)}. \tag{5.1}$$

Then, $K_N(p,q)$ is a *reproducing kernel* in the following sense:

We shall consider the family of all the functions, for arbitrary complex numbers $\{C_n\}_{n=0}^N$

$$F(p) = \sum_{n=0}^{N} C_n U_n(p) \tag{5.2}$$

and we introduce the norm

$$\|F\|^2 = \sum_{n=0}^{N} |C_n|^2. \tag{5.3}$$

The function space forms a Hilbert space $H_{K_N}(E)$ determined by the kernel $K_N(p,q)$ with the inner product induced from the norm (5.3), as usual. Then, we note that, for any $y \in E$

$$K_N(\cdot, q) \in H_{K_N}(E) \tag{5.4}$$

and for any $F \in H_{K_N}(E)$ and for any $q \in E$

$$F(q) = (F(\cdot), K_N(\cdot, q))_{H_{K_N}(E)} = \sum_{n=0}^{N} C_n U_n(q). \tag{5.5}$$

The properties (5.4) and (5.5) are called a *reproducing property* of the kernel $K_N(p,q)$ for the Hilbert space $H_{K_N}(E)$. This is an essence of reproducing kernel and reproducing kernel Hilbert space. This situation may be considered as the prototype example in Proposition 4.1. Recall the Pythagorean theorem as the simplest case. In particular, consider the cases $N = 0, 1, 2$.

We wish to introduce a preHilbert space by

$$H_{K_\infty} := \bigcup_{N \geq 0} H_{K_N}(E).$$

For any $F \in H_{K_\infty}$, there exists a space $H_{K_M}(E)$ containing the function F for some $M \geq 0$. Then, for any N such that $M < N$,

$$H_{K_M}(E) \subset H_{K_N}(E)$$

and, for the function $F \in H_{K_M}$,

$$\|F\|_{H_{K_M}(E)} = \|F\|_{H_{K_N}(E)}.$$

Therefore, there exists the limit:

$$\|F\|_{H_{K_\infty}} := \lim_{N \to \infty} \|F\|_{H_{K_N}(E)}.$$

Denote by H_∞ the completion of H_{K_∞} with respect to this norm. Then, we obtain:

Proposition 5.1. *Under the above conditions, for any function $F \in H_\infty$ and for F_N defined by*

$$F_N(p) = \langle F, K_N(\cdot, p) \rangle_{H_\infty},$$

$F_N \in H_{K_N}(E)$ for all $N > 0$, and as $N \to \infty$, $F_N \to F$ in the topology of H_∞.

Proposition 5.1 may be looked as a reproducing kernel Hilbert space H_∞ in the natural topology and in the sense of Proposition 5.1 the reproducing property may be written as follows:

$$F(p) = \langle F, K_\infty(\cdot, p)\rangle_{H_\infty},$$

with

$$K_\infty(\cdot, p) \equiv \lim_{N\to\infty} K_N(\cdot, p) = \sum_{n=0}^{\infty} U_n(\cdot)\overline{U_n(p)}. \tag{5.6}$$

Here *the limit does, in general, not need to exist*, however, the series are non-decreasing, in the sense: for any $N > M$, $K_N(q,p) - K_M(q,p)$ is a poisitive definite quadratic form function.

The function (5.6) may be looked a *generalized Delta function.*

Any reproducing kernel (separable case) may be considered as the form (5.6) by arbitrary linear independent functions $\{U_n(p)\}$ on an abstract set E, here, the sum does not need to converge. Furthermore, the property of linear independent is not essential.

This is why we proposed its form as a symbol of reproducing kernels in our new journal.

The completion H_∞ may be found, in concrete cases, from the realization of the spaces $H_{K_N}(E)$.

The typical case is that the family $\{U_n(p)\}_{n=0}^{\infty}$ is a complete orthonormal system in a Hilbert space with the norm

$$\|F\|^2 = \int_E |F(p)|^2 dm(p) \tag{5.7}$$

with a dm measurable set E in the usual form $L_2(E, dm)$. Then, the functions and the norm in the reproducing kernel Hilbert space are realized by this norm and the completion of the space $H_{K_\infty}(E)$ is given by this Hilbert space with the norm (5.7).

For any separable Hilbert space comprising functions, there exists a complete orthonormal system, and so, by our generalized sense, for the Hilbert space there exists an approximating reproducing kernel Hilbert spaces and therefore, the Hilbert space is the generalized reproducing kernel Hilbert space in this sense.

This will mean that *we were able to extend the classical reproducing kernels* ([3, 8, 55]), beautifully and completely.

For a positive definite function, non-symmetric expansions containing integral expressions were examined from the viewpoint of non-harmonic Fourier series. This leads to the concept of non-harmonic integral transforms. See Chapter 7 of [54] for the details.

6. PROBABILITY THEORY AND SUPPORT VECTOR MACHINES

For the linear mapping (4.1), to consider the kernel form (4.2) is essentially important, meanwhile any reproducing kernel is given in the form (4.2) and the form will be appeared naturally in different theories.

Kolmogorov factorization theorem (Kolmogorov 1961) gives, conversely for any positive definite quadratic form function $K(p, q)$, a factorization representation

(4.2) by constructing a Hilbert space $\mathcal{H}$ and a Hilbert $\mathcal{H}$-valued function $\mathbf{h}(p)$ on E. This important result was interestingly derived from the theory of stochastic theory independing of the theory of reproducing kernels. This property is essentially important when we consider a general convolution operator and various operators among abstract Hilbert spaces. In particular, among abstract Hilbert spaces, we can introduce various operators like products, sums and differentials. The typical concept is the idea of convolution. See (Saitoh 1999, [62]) for the details.

Let $(\Omega, \mathcal{B}, P)$ be a probability space and $L_2(\Omega, \mathcal{B}, P)$ the Hilbert space of composing of the second order random variables on Ω with the inner product $E(X\overline{Y})$. Let $X(t)$, t on a set T, be a second order stochastic process defined on the probability space $(\Omega, \mathcal{B}, P)$. For the mean value function as $m(t) = E(X(t))$, the second moment function

$$R(t, s) = E(X(t)\overline{Y(s)}) \tag{6.1}$$

and the covariance function

$$K(t, s) = E((X(t) - m(t))\overline{(Y(s) - m(s))}) \tag{6.2}$$

are positive definite quadratic form functions on Ω and so, the both theories of stochastic processes and reproducing kernels have a fundamental relationship. A typical result is the *Loéve's theorem*: The Hilbert space $H(X)$ generated by the process $X(t)$, t on a set T with the covariance function $R(t, s)$ is *congruent* to the reproducing kernel Hilbert space admitting the kernel $R(t, s)$.

The support vector machine is a powerful computational method for solving learning and function estimating problems such that pattern recognition, density and regression estimation and operator inversion. See (Vapnik [68]) for the details.

From some data input space E we consider a general non-linear mapping to a feature space F that is a pre-Hilbert space with the inner product $(\cdot, \cdot)_F$:

$$\Phi : E \longrightarrow F; \qquad x \longrightarrow \Phi(x). \tag{6.3}$$

Then, we form the positive definite quadratic form function

$$K(x, y) = (\Phi(x), \Phi(y))_F. \tag{6.4}$$

The important point of this method is that we can apply this kernel to the problem of construction of the optimal hyperplanes in the space F not by using the explicit values of the transformed data $\Phi(x)$. See (Berlinet and Thomas-Agnan [10, 11]) for the basic books and their references.

A new method is developing as *kernel method*:

For the transform of the data in the probability space $(\Omega, \mathcal{B}, P)$ for a reproducing kernel Hilbert space H_K admitting a kernel on Ω:

$$\Psi : \Omega \longrightarrow (\cdot, \cdot)_F; \qquad x \longrightarrow K(\cdot, x), \tag{6.5}$$

the theory of reproducing kernels may be applied to the probability problems on the space $(\Omega, \mathcal{B}, P)$.

On the whole space $\mathbb{R}^m$ the following kernels are typical:

(1) The usual inner product is given by $k(x_1, x_2) = x_1^T x_2$.
(2) For $c \geq 0$ and for a positive integer d

$$k_{d,c}^{\text{poly}}(x_1, x_2) = (x_1^T x_2 + c)^d \quad x_1, x_2 \in \mathbb{R}^m.$$

(3) The Gauss kernel, for $\sigma > 0$

$$k_\sigma^G(x_1, x_2) = \exp\Big(-\frac{|x_1 - x_2|^2}{2\sigma^2}\Big) \quad x_1, x_2 \in \mathbb{R}^m.$$

See the basic references (Fukumizu et al [32, 33]) and their references.

The Dirac delta function and the Green functions are a family of reproducing kernels, and orthonormal systems and reproducing kernels are the basic tool of *quantum mechanics; Coherent States*. See the general survey article (Vourdas [69]).

7. RANDOM FIELDS ESTIMATIONS

We now consider the following situation

(1) X is a set,
(2) $(\Omega, \mathcal{F}, P)$ is a probability space.

We assume that the random field is of the form

$$u(x) = s(x) + n(x) \quad (x \in X), \tag{7.1}$$

where, for each $x \in X$, $s(x) = s(x; \cdot) : \Omega \to \mathbb{R}$ is the useful signal and $n(x) = n(x; \cdot) : \Omega \to \mathbb{R}$ is a noise. Note that $s(x)$ and $n(x)$ are not necessarily independent for each $x \in X$. Without loss of generality, we can assume that the mean values of $u(x)$ and $n(x)$ are zero. We assume that the covariance functions

$$R(x, y) = E[u(x)u(y)] \quad (x, y \in X) \tag{7.2}$$

and the information

$$f(x, y) = E[u(x)s(y)] \quad (x, y \in X) \tag{7.3}$$

are known.

Here and below, we equiv X with the structure of the measure space: let (X, dm) be a measure space.

In addition, we shall consider the general form of a linear estimation $\hat{u}$ of u in the form

$$\hat{u}(x) = \int_X u(y)h(x, y)dm(y) = \langle u, h(x, \cdot)\rangle_{L^2(X,dm)} \tag{7.4}$$

for an $L^2(X, dm)$ space and for a function $h(x, \cdot)$ belonging to $L^2(X, dm)$ for any fixed $x \in E$. For the desired information $As : X \times \Omega \to \mathbb{R}$, which satisfies

$$A(ks) = kAs, \ A(s_1 + s_2) = As_1 + As_2$$

for all $s, s_1, s_2 : X \times \Omega \to \mathbb{R}$ and $k \in \mathbb{C}$, we wish to determine the function $h(x, t)$ attaining

$$\inf\{E[(\hat{u}(x) - As(x))^2] \, : \, \hat{u} \text{ is given by (7.4) and } h(x, \cdot) \in L^2(X, dm)\} \tag{7.5}$$

which gives the minimum of the variance by the least squares method.

Many topics in filtering and estimation theory in signal and image processing, underwater acoustics, geophysics, optical filtering, etc., which were initiated by

N. Wiener (1894–1964), will be presented in this framework. Then, we see that the linear transform $h(x,t)$ is given by the integral equation

$$\int_X R(x',y)h(x,y)dm(y) = f(x',x). \tag{7.6}$$

Therefore, our random fields estimation problems will be reduced to finding the inversion formula

$$f \mapsto h \tag{7.7}$$

in our framework. So, our general method for integral transforms will be applied to these problems. For this situation and for other topics on the inversion formulas, see the textbook [49], which handles the topic very concisely.

8. DIFFERENTIAL EQUATIONS AND INTEGRAL EQUATIONS

We will refer to the very important connection with very general linear integro-differential equations. For the classical integral transform method, the coefficients of differential equations are restricted strongly and essentially, however, for the method introduced here, called the **backward transform method**, surprisingly enough, we do not need any restriction for the coefficients, essentially.

We consider the following extremely general, however, linear integro-differential equations: For open intervals T and E, consider a linear integro-differential equation:

$$a_0(t)F(t) + a_1(t)F'(t) + \cdots + a_n(t)F^{(n)}(t) + \int_T F(\xi)\overline{h(\xi,t)}dm(\xi) = f(t) \tag{8.1}$$

on E, where

$$h(\cdot,t) \in L^2(T,dm) \quad \text{for} \quad t \in E, \tag{8.2}$$

and $\{a_j\}_{j=0}^n$ are *arbitrary* complex-valued functions on E.

From the form (8.1), we shall assume that F belongs to some reproducing kernel Hilbert space $H_{\mathbb{K}}$ on E satisfying

$$\mathbb{K}(t,t') = \int_{\hat{E}} \hat{h}(\hat{\xi},t')\overline{\hat{h}(\hat{\xi},t)}d\hat{m}(\hat{\xi}), \quad \text{on} \quad E \times E \tag{8.3}$$

and the system $\{\hat{h}(\cdot,t); t \in E\}$ is complete in $L^2(\hat{E},d\hat{m})$. Then, any member $F \in H_{\mathbb{K}}(E)$ is expressible in the form

$$F(t) = \int_{\hat{E}} \hat{F}(\hat{\xi})\overline{\hat{h}(\hat{\xi},t)}d\hat{m}(\hat{\xi}) \quad (t \in E) \tag{8.4}$$

and we have the isometric identity

$$\|F\|_{H_{\mathbb{K}}} = \sqrt{\int_{\hat{E}} |\hat{F}(\hat{\xi})|^2 d\hat{m}(\hat{\xi})}. \tag{8.5}$$

From (8.1), we assume furthermore that

$$\frac{\partial^{j+j'}\mathbb{K}}{\partial t^j \partial t'^{j'}}(t,t') \quad (j,j' = 0,1,2,\ldots,n) \quad \text{on} \quad E \times E \tag{8.6}$$

are continuous. Then, we see that any member F of $H_{\mathbb{K}}$ belongs to the $C^n(T)$-class and we have the expression

$$F^{(j)}(t) = \int_{\hat{E}} \hat{F}(\hat{\xi}) \overline{\frac{\partial^j \hat{h}}{\partial t^j}(\hat{\xi}, t)} d\hat{m}(\hat{\xi}), \quad \text{on} \quad E. \tag{8.7}$$

Proposition 8.1. *Assume*

$$\int_E \mathbb{K}(t,t) dm(t) < \infty.$$

Then we have

$$f(t) = \int_{\hat{E}} \hat{F}(\hat{\xi}) \left\{ a_0(t) \overline{\hat{h}(\hat{\xi}, t)} + \cdots + a_n(t) \overline{\partial_t^n \hat{h}(\hat{\xi}, t)} \right\} d\hat{m}(\hat{\xi})$$
$$+ \int_{\hat{E}} \hat{F}(\hat{\xi}) \left\{ \int_T \overline{\hat{h}(\hat{\xi}, \eta)} \cdot \overline{h(\eta, t)} dm(\eta) \right\} d\hat{m}(\hat{\xi})$$

for all $t \in E$.

Our procedure implies that the integro-differential equation (8.1) can be transformed to the Fredholm integral equation of the first kind.

The difficulty of solving the integro-differential equation (8.1) with variable coefficients will be transformed to that of the complicated form in the integral kernel in (8.1). However, we should note that for integro-differential equations (8.1) with arbitrary functions as the coefficients we can deal with them.

9. GREEN'S FUNCTIONS, DELTA FUNCTION, DIFFERENTIAL EQUATIONS AND REPRODUCING KERNELS.

In order to see some essence, we shall consider a bounded regular domain D whose boundary is made up of a finite number of analytic Jordan curves.

Consider a unit point mass distribution $\delta(p,q)$ (called **Dirac's delta function**) at $q \in D$, this will mean that for example, for any continuous function f on D, we have

$$f(q) = \int_D f(p) \delta(p,q) dp. \tag{9.1}$$

We will be able to consider such representation for various function spaces.

Meanwhile, suppose that a solution $G(p,q)$ for some linear (differential) operator L on some function space on D is given by the equation, symbolically, for any fixed $q \in D$

$$LG(p,q) = \delta(p,q) \tag{9.2}$$

whose identity is valid on D except for the point $q \in D$. When G depends only on the distance $|p-q|$, then such function $G(p,q)$ will be called a **fundamental solution** for the operator L and further some boundary conditions are satisfied, then such a function $G(p,q)$ will be called a **Green's function** for the operator L satisfying the imposed boundary conditions.

In general, the function $G(p,q)$ may be understood as the impulse response for the system L, physically.

For the adjoint operator L^* of L, we shall consider the self-adjoint operator L^*L and its Green's function $G(p,q)$ satisfying

$$L^*LG(p,q) = \delta(p,q) \tag{9.3}$$

in the formal level, whose identity is valid on D except for the point $q \in D$. Then, from (9.1) we obtain the representation

$$f(q) = \int_D f(p)L^*LG(p,q)dp. \tag{9.4}$$

Then, we can obtain the representation symbolically, by using the Green-Stokes formula,

$$f(q) = \int_D Lf(p)LG(p,q)dp + \text{ some boundary integrals.} \tag{9.5}$$

If the boundary integrals are zero, then we have

$$f(q) = \int_D Lf(p)LG(p,q)dp. \tag{9.6}$$

If the function space is made up of all f satisfying

$$\int_D |Lf(p)|^2 dp < \infty, \tag{9.7}$$

then the space forms a Hilbert space and if the function $G(p,q)$ is a usual function on D belonging to this Hilbert space, then the function $G(p,q)$ will represent the reproducing property for the Hilbert space.

Indeed, we can find many and many cases satisfying these properties. Following these frameworks, we can see the basic relationship among Dirac's delta function, Green's functions and reproducing kernels.

We will be able to consider Dirac's delta function, Green's functions and reproducing kernels as a family in the above senses.

We shall look the prototype example that shows clearly the above general idea.

For fixed $a, b > 0$,

$$G(s,t) \equiv \frac{1}{2ab} \exp\left(-\frac{b}{a}|s-t|\right) \quad (s,t \in \mathbb{R}) \tag{9.8}$$

is the reproducing kernel for the Hilbert space $H_G(\mathbb{R}) = W^{1,2}(\mathbb{R})$ equipped with the norm

$$\|f\|_{W^{1,2}(\mathbb{R})} = \sqrt{\int_{\mathbb{R}} (a^2|f'(x)|^2 + b^2|f(x)|^2)\,dx}. \tag{9.9}$$

The reproducing property may be looked as follows:

$$\begin{aligned}
&\int_{\mathbb{R}} a^2 f'(s) \frac{dG_t}{ds}(s) + b^2 f(s) G(s,t)\, ds \\
&= \int_{-\infty}^{t} f'(s) \frac{dG_t}{ds}(s) + b^2 f(s) G(s,t)\, ds + \int_{t}^{\infty} a^2 f'(s) \frac{dG_t}{ds}(s) + b^2 f(s) G(s,t)\, ds \\
&= \frac{1}{2} \int_{-\infty}^{t} \left(\frac{b}{a} f(s) - f'(s) \right) \exp\left(\frac{b}{a}(s-t) \right) ds \\
&\quad + \frac{1}{2} \int_{t}^{\infty} \left(f'(s) + \frac{b}{a} f(s) \right) \exp\left(\frac{b}{a}(t-s) \right) ds \\
&= \frac{1}{2} \left[f(s) \exp\left(\frac{b}{a}(s-t) \right) \right]_{-\infty}^{t} - \frac{1}{2} \left[f(s) \exp\left(\frac{b}{a}(t-s) \right) \right]_{t}^{\infty} = f(t).
\end{aligned}$$

However, this will mean that

$$\int_{\mathbb{R}} a^2 f'(s) \frac{dG_t}{ds}(s) + b^2 f(s) G(s,t)\, ds = \int_{\mathbb{R}} f(s) \left(-a^2 \frac{d^2 G_t}{ds^2}(s) + b^2 G_t(s) \right) ds = f(t);$$

that is, $G(s,t)$ is the Green's function satisfying the differential equation

$$-a^2 y''(s) + b^2 y(s) = \delta(s,t)$$

on $\mathbb{R}$ satisfying the null property at ∞.

For some general and deep relationship between the Green's functions and the reproducing kernels, see [4].

10. GENERAL NONLINEAR TRANSFORMS AND REPRODUCING KERNELS

In 1976, the generalized isoperimetric inequality was obtained by the application of the general theory of reproducing kernels in [52]: For a bounded regular region G in the complex $z = x + iy$ plane whose boundary is surrounded by a finite number of analytic Jordan curves and for any analytic functions φ and ψ on $\overline{G} = G \cup \partial G$,

$$\frac{1}{\pi} \iint_G |\varphi(z)\psi(z)|^2 dxdy \leq \left(\frac{1}{2\pi} \oint_{\partial G} |\varphi(z)|^2 |dz| \right) \left(\frac{1}{2\pi} \oint_{\partial G} |\psi(z)|^2 |dz| \right). \quad (10.1)$$

In order to prove (10.1), we have to use the long historical results of the following great mathematicians:

> G. F. B. Riemann (1826–1866); F. Klein (1849–1925); S. Bergman; G. Szegö; Z. Nehari; M. M. Schiffer; P. R. Garabedian; D. A. Hejhal (1972, thesis).

In particular, a profound result of D. A. Hejhal, which establishes the fundamental relationship between the Bergman and the Szegö reproducing kernels of G [36], must be applied. Furthermore, we must use the general theory of reproducing kernels by N. Aronszajn [3] described in 1950. These circumstances remain at this moment since the paper [52] was published in 1979.

Meanwhile, the main ingredient in the paper was to determine the equality case in the above inequality; there, some deep hard analysis was used in the function theory of one complex variable, stated in the above line. A very deep and general proof appeared 26 years later in A. Yamada. He gave a very general framework for such equality problems in the tensor products of reproducing kernel Hilbert spaces. We can see his deep theory in [62], Section A.1.

Now, we see an important meaning or application of the inequality (10.1); that is, when we fix any member ψ of $H_2(G)$, the multiplication operator

$$\varphi \longmapsto \varphi(z)\psi(z), \tag{10.2}$$

on $H_2(G)$ to the Bergman space is bounded. Therefore, by the general theory for general fractional functions, we can consider the generalized fractional functions: for any Bergman function $f(z)$ on the domain G

$$\frac{f(z)}{\psi(z)}, \tag{10.3}$$

at least in the sense of Tikhonov; that is, we can consider the best approximation problem for the functions $\psi(z)^{-1}f(z)$ by the functions $H_2(G)$. See [18] for more detailed results.

This paper was a milestone in the development of the theory of reproducing kernels. Starting of the paper, various applications of the general theory of reproducing kernels were developed. See also [55] for the details. It seems that the general theory of reproducing kernels was, in a strict sense, not active in the theory of concrete reproducing kernels until the publication of the paper. Indeed, after the publication of the paper, various fundamental norm inequalities containing quadratic norm inequalities in matrices were derived. Furthermore, a general idea for linear transforms essentially by using the general theory of reproducing kernels, stated here was obtained.

Surprisingly enough, since 40 years later after publication of [52], for some open question proposed there, an entirely unexpected partial solution was published in [34] that is an entirely new result.

We assume that the domain D is a regular domain surrounded by a finite number of analytic Jordan curves. Let $G(z,t)$ be the Green's function on D such that $G(z,t) + \log|z-t|$ is analytic on $D \times D$ with pole at a fixed point t of D. Let $\partial/\partial\nu$ denote the inner normal derivative on the boundary ∂D and $\partial G(z,t)/\partial\nu$ is positive and real analytic on the boundary ∂D. Then,

Proposition 10.1. *For any fixed $t \in D$ and for any fixed analytic function f on D which is continuous on $D \cup \partial D$, the identity*

$$\lim_{r\to 1-0} \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f(z)|^2 dxdy$$

$$= \frac{1}{2} \int_{\partial D} |f(z)|^2 (\partial G(z,t)/\partial\nu)^{-1} |dz|$$

holds.

The following inequalities seem to be interesting in its own sense:

Theorem 10.2. *For any given $\epsilon > 0$ and for any fixed analytic function $f(z)$ on $D \cup \partial D$, there exists $r : (0 < r < 1)$ satisfying the inequality*

$$\int\int_D |f'(z)|^2 dxdy - \epsilon$$
$$\leq \frac{1}{1-r} \int\int_{\{e^{-2G(z,t)} \geq r\}} |f'(z)|^2 dxdy.$$

This inequality may be looked as an isoperimetric inequality, because the Dirichlet integral on a domain is estimated (restricted) by the Dirichlet integral on some small boundary neighborhood of the domain. Here, the neighborhood size and estimation are stated by the level curve of the Green function, precisely.

Even the case of the identity function $f(z) = z$, we can enjoy the senses of the estimation and the result.

For any analytic function $f(z)$ on $D \cup \partial D$, we have the inequality

$$\int\int_D |f'(z)|^2 dxdy$$
$$\leq \frac{1}{2} \int_{\partial D} |f'(z)|^2 (\partial G(z,t)/\partial \nu)^{-1} |dz|.$$

This result was obtained from some complicated relations among reproducing kernels in ([53]). The equality problem in the inequality was also established; that is, equality holds if and only if the domain is simply-connected and the function $f'(z)$ is expressible in the form $CK(z, \overline{t})$ for the Bergman reproducing kernel $K(z, \overline{u})$ on the domain D and for a constant C. Theorem 10.2 is derived from this inequality and Proposition 10.1.

Of course, the image identification of nonlinear transforms of a reproducing kernel Hilbert space is difficult, however, it will be very important that we can find the natural image space as a reproducing kernel Hilbert space that contains the images and we can obtain the natural norm inequalities as in the following

Proposition 10.3. *Let K be a positive definite quadratic form function on E. Suppose that we are given a sequence of functions $\{d_n\}_{n=0}^{\infty}$ on E satisfying*

$$\sum_{n=0}^{\infty} |d_n(p)|^2 K(p,p)^n < \infty \tag{10.4}$$

for all $p \in E$. Define

$$K_d(p,q) \equiv \sum_{n=0}^{\infty} d_n(p)\overline{d_n(q)} K(p,q)^n \quad (p, q \in E). \tag{10.5}$$

If $f \in H_K(E)$ satisfies

$$\sum_{n=0}^{\infty} (\|f\|_{H_K(E)})^{2n} < \infty, \tag{10.6}$$

or equivalently $\|f\|_{H_K(E)} < 1$, *then the sum* $\Phi f(p) \equiv \sum_{n=0}^{\infty} d_n(p) f(p)^n$ *converges absolutely on* E *in* $H_{K_d}(E)$. *Furthermore,*

$$\|\Phi f\|_{H_{K_d}(E)}{}^2 \leq \sum_{n=0}^{\infty} \left(\|f\|_{H_K(E)}\right)^{2n}. \tag{10.7}$$

The method has proved to be very important for applications such as identifications of nonlinear systems. In particular, note that for a very general nonlinear mapping like the images of nonlinear differential operators, we can find the method how to find the properties of the images.

In the theory of nonlinear partial differential equations, we encounter nonlinear transforms, for example, for $u = u(x,t)$, the KDV equation

$$u \mapsto u_t + 6uu_x + u_{xxx} \tag{10.8}$$

and

$$u \mapsto u_{tt} - u_{xx} + m^2 \sin u, \tag{10.9}$$

where $m > 0$ is a constant. For such nonlinear transforms we show how to find the properties of the operators.

In order simply to state the result, we assume that I is an open interval on $\mathbb{R}$. Then, for the smoothness of $H_K(I)$, note that if

$$\frac{\partial^{(j+j')} K}{\partial x^j \partial y^{j'}}(x,y) \tag{10.10}$$

are continuously differentiable on $I \times I$, then for any member f of $H_K(I)$, $f^{(j)} (j \leq n)$ are also continuously differentiable on I and we have

$$f^{(n)} \in H_{K^{n,n}(I)} \tag{10.11}$$

and

$$\|f^{(n)}\|_{K^{n,n}(I)} \leq \|f\|_{H_K(I)}, \tag{10.12}$$

for the reproducing kernel Hilbert space $H_{K^{n,n}}(I)$ admitting the reproducing kernel

$$K^{n,n}(x,y) = \frac{\partial^{2n} K}{\partial x^n \partial y^n}(x,y) \quad \text{on} \quad I \tag{10.13}$$

as the elementary property. Hence, for example, in the nonlinear transform

$$\psi : f \in H_K(I) \mapsto h_1(x) f''(x) + h_2(x) f'(x)^2 + h_3(x)|f(x)|^2 \tag{10.14}$$

for any complex-valued functions $\{h_j\}_{j=1}^3$ on I, the images $\psi(f)$ belong to the reproducing kernel Hilbert space $H_{\psi^+(K)}(I)$ admitting the reproducing kernel

$$\psi^+(K(x,y)) = h_1(x)\overline{h_1(y)}K^{2,2}(x,y) + h_2(x)\overline{h_2(y)}K^{1,1}(x,y)^2 + h_3(x)\overline{h_3(y)}|K(x,y)|^2$$

for $x, y \in I$, and, we obtain the inequality

$$\|\psi(f)\|_{H_{\psi^+(K)}(I)}{}^2 \leq \|f\|_{H_K(I)}{}^2 + 2\|f\|_{H_K(I)}{}^4. \tag{10.15}$$

It is worth noting that the right-hand side does not depend on $\{h_j\}_{j=1}^3$.

In some general linear transform of Hilbert spaces we could get essentially isometries between the input and the output function spaces. However, in nonlinear transforms, we get norm inequalities, essentially.

Among many concrete norm inequalities, the most beautiful and simple one is the following one.

Note that

$$K(x,y) = \min(x,y) \quad (0 \le x, y < \infty) \tag{10.16}$$

is the reproducing kernel for the Hilbert space H_0 consisting of all real-valued and absolutely continuous functions $f(x)$ on $[0,\infty)$ such that $f(0) = 0$ and that

$$\|f\|_{H_0} = \sqrt{\int_0^\infty |f'(x)|^2 dx} < \infty, \tag{10.17}$$

as we see directly.

Let us set

$$\varphi_N(t) \equiv \begin{cases} \dfrac{t(1-t^N)}{1-t} & t \neq 1, \\ N & t = 1. \end{cases}$$

Then,

$$\mathbf{K_N}(x,y) = \sum_{n=1}^{N} K(x,y)^n = \min\{\varphi_N(x), \varphi_N(y)\} \quad (x, y \in [0,\infty)) \tag{10.18}$$

is the reproducing kernel for the Hilbert space $H_{0,N}$ consisting of all real-valued and absolutely continuous functions f on $[0,\infty)$ such that $f(0) = 0$ and that f has finite norms

$$\|f\|_{\mathbf{K_N}} = \sqrt{\int_0^\infty f'(x)^2 \left(\frac{x(1-x^N)}{1-x}\right)'^{-1} dx} < \infty. \tag{10.19}$$

Hence, for the nonlinear transform of $f \in H_0$

$$\varphi(f)(x) = \sum_{n=1}^{N} f(x)^n = \frac{f(x)(1-f(x)^N)}{1-f(x)}, \tag{10.20}$$

we have the inequality

$$\begin{aligned} \|\varphi(f)\|_{H_{\varphi(\mathbf{K}_N)}}{}^2 &= \int_0^\infty \left|\left(\frac{f(x)(1-f(x)^N)}{1-f(x)}\right)'\right|^2 \left|\left(\frac{x(1-x^N)}{1-x}\right)'\right|^{-1} dx \\ &\le \frac{a(1-a^N)}{1-a}, \end{aligned} \tag{10.21}$$

where

$$a = a_f = \int_0^1 f'(x)^2 dx. \tag{10.22}$$

In particular, for $f \in H_0$ with $a_f \in (0,1)$ we have the inequality, by letting $N \to \infty$

$$\int_0^1 \left(\frac{f(x)}{1-f(x)}\right)'^2 (1-x)^2 dx \le \frac{a}{1-a}. \tag{10.23}$$

We thus obtain:

Proposition 10.4. *For a real-valued function* $f \in \mathrm{AC}[0,1]$ *with* $f(0) = 0$ *and* $\int_0^1 f'(x)^2 dx < 1$*, we have*

$$\int_0^1 \Big(\frac{f(x)}{1-f(x)}\Big)'^2 (1-x)^2 dx \leq \frac{\int_0^1 f'^2(x)\,dx}{1-\int_0^1 f'^2(x)\,dx}. \tag{10.24}$$

Equality holds in (10.24), if f is of form $f(x) = \min\{x, y\},\ x \in [0,1]$ for some $y \in [0,1)$. But the equality condition was very difficult to determine. To solve the equality problems for the norm inequalities in nonlinear transforms seem to be the most deep theory in the general and abstract theory of reproducing kernels. It was discussed deeply by A. Yamada for a general situation containing many concrete norm inequalities, see [62], Appendix A.1.

11. EIGENFUNCTIONS, INITIAL VALUE PROBLEMS AND REPRODUCUNG KERNENLS

For some general linear operator L_x (and differential operator ∂_t), for some function space on a certain domain (to be specified later on), we shall consider the problem

$$(\partial_t + L_x)u_f(t,x) = 0, \quad t > 0, \tag{11.1}$$

for an unknown u_f satisfying the initial value condition

$$u_f(0,x) = f(x). \tag{11.2}$$

For such a very global problem, a general method which includes the analysis of the existence and construction of the solution of that type of initial value problem by using the theory of reproducing kernels may be considered. Furthermore, the method will have the power completely characterize the solutions under each specific conditions.

One of the basic procedures in the method is to use some eigenvalues λ on a set I, and eigenfunctions W_λ satisfying

$$L_x W_\lambda = \lambda W_\lambda.$$

The case of discrete eigenvalues may be dealt with similarly and so we shall assume that the eigenvalues are continuous on an interval I for $\lambda > 0$. In this way, we note that the functions

$$\exp(-\lambda t)W_\lambda(x) \tag{11.3}$$

are the solutions of the operator equation

$$(\partial_t + L_x)u(t,x) = 0. \tag{11.4}$$

We shall consider some general solution of (11.4) by a suitable sum of the elements in (11.3). In order to consider a convenient sum, we shall use the following kernel form, with a continuous non-negative weight function ρ over the interval I,

$$\int_I \exp(-\lambda t)W_\lambda(x)W_\lambda(y)\rho(\lambda)d\lambda \tag{11.5}$$

(where we are naturally considering the integral with absolutely convergence for the kernel form). Moreover, here we assume λ to be real-valued and also the eigenfunctions $W_\lambda(x)$ are real-valued. Then, fully general solutions of the equation (11.4) may be represented in the integral form

$$u(t,x) = \int_I \exp(-\lambda t) W_\lambda(x) F(\lambda)\rho(\lambda) d\lambda \tag{11.6}$$

for the functions F satisfying

$$\int_I \exp(-\lambda t)|F(\lambda)|^2 \rho(\lambda) d\lambda < \infty. \tag{11.7}$$

Therefore, the solution $u(t,x)$ of (11.4) satisfying the initial condition

$$u(0,x) = F(x) \tag{11.8}$$

will be obtained by taking $t \to 0$ in (11.6). However, this point will be very delicate and we will need to consider some deep and intricate structure. Here, (11.5) is a reproducing kernel and in order to analyze in detail the strategy above, we will need the theory of reproducing kernels. In particular, in order to construct certain natural solutions of (11.5) we will need a new framework and function space.

In order to analyze the integral transform (11.6) and in order to set the basic background for our purpose, we will need the essence of the theory of reproducing kernels.

For this purpose, we will assume that I is a positive interval, $\lambda > 0$, and that this parameter λ represents the eigenvalues satisfying

$$L_x \overline{h_\lambda} = \lambda \overline{h_\lambda}, \quad x \in E, \quad \lambda \in I. \tag{11.9}$$

Here, $\overline{h_\lambda}$ is the eigenfunction and in order to set our notation in a consistent way, we put the complex conjugate there.

We form the reproducing kernel

$$K_t(x,y) = \int_I \exp(-\lambda t) h_\lambda(y) \overline{h_\lambda}(x)\, dm(\lambda), \quad t > 0, \tag{11.10}$$

and consider the reproducing kernel Hilbert space $H_{K_t}(E)$ admitting the kernel K_t. In particular, note that, for $K_0(x,y) = K(x,y)$

$$(K_t)_y \in H_K(E), \quad y \in E.$$

Then, we have the following main result

Proposition 11.1. *For any element $f \in H_K(E)$, the solution $u_f(t,x)$ of the initial value problem* (11.1)–(11.2) *exists and it is given by:*

$$u_f(t,x) = \langle f, (K_t)_x \rangle_{H_K(E)}. \tag{11.11}$$

In fact, the solution (11.11) *satisfies* (11.2) *in the sense*

$$\lim_{t \to +0} u_f(t,x) = \lim_{t \to +0} \langle f, K_t(\cdot,x)\rangle_{H_K(E)} = \langle f, K_x \rangle_{H_K(E)} = f(x), \tag{11.12}$$

whose existence is ensured and the limit is given in the sense of uniformly convergence on any subset of E such that $K(x,x)$ is bounded.

The uniqueness property of the initial value problem is depending on the completeness of the family of functions

$$\{(K_t)_x; x \in E\} \tag{11.13}$$

in $H_K(E)$.

For some concrete example and practical applications, see the book [62], Chapter 8.7.

In the Proposition, the complete property of the solutions $u_f(t,x)$ of (11.1)–(11.2) satisfying the initial value f may be derived by the reproducing kernel Hilbert space admitting the kernel

$$k(x,t;y,\tau) := \langle (K_\tau)_y, (K_t)_x \rangle_{H_K(E)}. \tag{11.14}$$

In the method, we see that the existence problem of the initial value problem is based on the eigenfunctions and we are constructing the desired solution satisfying the initial condition. In view of this, for a larger knowledge for the eigenfunctions we can consider a more general initial value problem. Furthermore, by considering the linear mapping of (11.11) with various situations, we will be able to obtain various inverse problems which may be described by looking for the initial values f from the various output data of $u_f(t,x)$.

We also would like to remark that in the stationary case of

$$L_x u(x) = 0, \tag{11.15}$$

the method is more simple and direct. We may consider the family of solutions $\overline{h_\lambda}$ such that

$$L_x \overline{h_\lambda} = 0. \tag{11.16}$$

Then, the general solutions are constructed by the integral form

$$u(x) = LF(x) \equiv \langle F, h_x \rangle_{L^2(I,dm)} = \int_I F(\lambda)\overline{h_\lambda}\, dm(\lambda), \quad x \in E, \tag{11.17}$$

for $F \in \mathcal{H} = L^2(I, dm)$.

The uniqueness property of the initial value problem follows directly from (11.11).

We will refer to how to look for various eigenfunctions.

First, from general solutions of homogeneous equations, we can look for eigenfunctions, of course.

For example, we consider modified Bessel differential equations

$$x^2 \frac{d^2y}{dx^2} + x\frac{dy}{dx} - (x^2 + \nu^2)y = 0 \tag{11.18}$$

which have as solutions the modified Bessel functions of second kind, i.e., $y = K_\nu$. Consequently, K_ν are eigenfunctions of the second order differential, i.e.

$$L_x K_\nu(x) = \nu^2 K_\nu(x) \tag{11.19}$$

with an eigenvalue ν^2, and where

$$L_x = x^2 \frac{d^2}{dx^2} + x\frac{d}{dx} - x^2. \tag{11.20}$$

Then, note that in the case that ν are positive reals, the functions

$$\exp(-\nu^2 t) K_\nu(x)$$

are the solutions of the operator equation

$$(\partial_t + L_x)\, u(t,x) = 0.$$

In addition, for pure imaginary values of ν the functions

$$\exp(\nu^2 t) K_\nu(x) \tag{11.21}$$

are the solutions of the operator equation

$$(\partial_t - L_x)\, u(t,x) = 0. \tag{11.22}$$

Furthermore, for a pure imaginary $\nu = i\tau$, we have as eigenvalues associated to the second order differential (11.20) $\nu^2 = -\tau^2$. For these eigenfunctions, we can discuss the detail results following the general theory.

For some concrete examples and practical applications, see the book [62], Chapter 8.7.

12. INVERSION FORMULAS

Consider the inversion in (4.1) formally, however, this idea will be very important for the general inversions as the central split and for very general discretization method in the theory of reproducing kernels.

Following the above general situation, let $\{\mathbf{v_j}\}$ be a complete orthonormal basis for $\mathcal{H}$. Then, for

$$v_j(p) = (\mathbf{v_j}, \mathbf{h}(p))_{\mathcal{H}},$$

$$\mathbf{h}(p) = \sum_j (\mathbf{h}(p), \mathbf{v_j})_{\mathcal{H}} \mathbf{v_j} = \sum_j \overline{v_j(p)} \mathbf{v_j}.$$

Hence, by setting

$$\overline{\mathbf{h}}(p) = \sum_j v_j(p) \mathbf{v_j},$$

$$\overline{\mathbf{h}}(\cdot) = \sum_j v_j(\cdot) \mathbf{v_j}.$$

Thus, define

$$(f, \overline{\mathbf{h}}(p))_{H_K} = \sum_j (f, v_j)_{H_K} \mathbf{v_j}.$$

For simplicity, write as follows:

$$H_K = H_K(E).$$

Then, formally, we obtain:

Proposition 12.1. *Assume that for* $f \in H_K$

$$(f, \overline{\mathbf{h}})_{H_K} \in \mathcal{H}$$

and for all $p \in E$,

$$(f, (\mathbf{h}(p), \mathbf{h}(\cdot))_{\mathcal{H}})_{H_K} = ((f, \overline{\mathbf{h}})_{H_K}, \mathbf{h}(p))_{\mathcal{H}}.$$

Then,

$$\|f\|_{H_K} \le \|(f, \overline{\mathbf{h}})_{H_K}\|_{\mathcal{H}}.$$

If $\{\mathbf{h}(p); p \in E\}$ *is complete in* $\mathcal{H}$*, then equality always holds.*

Furthermore, if:

$$(\mathbf{f}_0, (f, \overline{\mathbf{h}})_{H_K})_{\mathcal{H}} = ((\mathbf{f}_0, \mathbf{h})_{\mathcal{H}}, f)_{H_K} \quad for \quad \mathbf{f}_0 \in N(L).$$

Then, for $\mathbf{f}^*$ *in (II) and (III)*

$$\mathbf{f}^* = (f, \overline{\mathbf{h}})_{H_K}.$$

In particular, note that the basic assumption $(f, \overline{\mathbf{h}})_{H_K} \in \mathcal{H}$ in Proposition 12.1, is, in general, not valid and very delicate for many analytical problems and so we need some delicate treatment for the inversion.

In order to derive a general inversion formula for (4.1) that is widely applicable in analysis, assume that the both Hilbert spaces $\mathcal{H}$ and H_K are given as $\mathcal{H} = L_2(T, dm), H_K \subset L_2(E, d\mu)$, on the sets T and E, respectively (assume that for $dm, d\mu$ measurable sets T, E, they are the Hilbert spaces comprising $dm, d\mu - L_2$ integrable complex-valued functions, respectively.) Therefore, consider the integral transform

$$f(p) = \int_T F(t)\overline{h(t,p)}dm(t). \tag{12.1}$$

Here, $h(t,p)$ is a function on $T \times E$, $h(\cdot, p) \in L_2(T, dm)$, and $F \in L_2(T, dm)$. The corresponding reproducing kernel for (4.2) is given by

$$K(p,q) = \int_T h(t,q)\overline{h(t,p)}dm(t) \quad \text{on} \quad E \times E.$$

The norm of the reproducing kernel Hilbert space H_K is represented as $L_2(E, d\mu)$.

Under these situations:

Proposition 12.2. *Assume that an approximating sequece* $\{E_N\}_{N=1}^{\infty}$ *of* E *satisfies (a)* $E_1 \subset E_2 \subset \cdots \subset \cdots$, *(b)* $\bigcup_{N=1}^{\infty} E_N = E$, *(c)* $\int_{E_N} K(p,p)d\mu(p) < \infty, \quad (N = 1, 2, ...)$.

Then, for $f \in H_K$ *satisfying* $\int_{E_N} f(p)h(t,p)d\mu(p) \in L_2(T, dm)$ *for any* N, *the sequence*

$$\left\{ \int_{E_N} f(p)h(t,p)d\mu(p) \right\}_{N=1}^{\infty} \tag{12.2}$$

converges to F^* *in (4.4) in Proposition 4.1 in the sense of* $L_2(T, dm)$ *norm.*

Practically for many cases, the assumptions in Proposition 12.2, will be satisfied automatically, and so Proposition 12.2 may be applied in many cases. Proposition 12.2 will give a new viewpoint and method for the Fredholm integral equation (12.1) of the first kind that is a fundamental integral equation. The method and solution has the following properties:

1) The use of the naturally determined reproducing kernel Hilbert space H_K which is determined by the integral kernel.

2) The solution is given in the sense of $\mathcal{H}$ norm convergence.

3) The solution (inverse) is given by $\boldsymbol{f}^*$ in Proposition 4.1.

4) For the ill-posed problem in (12.1), the solution is given as a well-posed solution.

This viewpoint is, however, a mathematical and theoritical one. Practically and analytically, first, the realization of the reproducing kernel is an essential problem. In practical and physical linear systems, the observation data will be *a finite number of data containing error or noises,* and so we will meet to various delicate problems numerically.

13. GENERALIZED INTEGRAL TRANSFORMS

The basic assumption here for the integral kernels is to belong to some Hilbert spaces. However, as a very typical integral transform, in the case of Fourier integral transform, the integral kernel does not belong to $L_2(\mathbf{R})$ and, however, we can establish the isometric identity and inversion formula in the space $L_2(\mathbf{R})$. Therefore, we need the natural extension of our idea.

When we consider the integral transform

$$LF(p) = \int_T F(\lambda)\overline{h(\lambda, p)}\, dm(\lambda), \quad p \in E \tag{13.1}$$

for $F \in \mathcal{H} = L^2(T, dm)$, indeed, the integral kernel $h(\lambda, p)$ does not need to belong to the space $L^2(T, dm)$ and with the very general assumptions that for any exhausion $\{T_t\}$ of T such that $T_t \subset T_{t'}$ for $t \leq t'$, $\bigcup_{t>0} T_t = T$,

$$h(\lambda, p) \quad \text{belongs to } L^2(T_t, dm) \text{ for any} \quad p \quad \text{of} \quad E$$

and

$$\{h(\lambda, p); p \in E\} \text{ is complete in } L^2(T_t, dm),$$

we can establish the isometric identity and inversion formula of the integral transform (13.1) by giving the natural interpretation of the integral transform (13.1), as in the Fourier transform by considering the generalized reproducing kernel $K(p, q)$

$$K_t(p, q) = \int_{T_t} h(t, q)\overline{h(t, p)} dm(t) \quad \text{on} \quad E \times E,$$

and

$$K(p, q) = \lim_{t \to \infty} K_t(p, q),$$

which diverges as in the delta function in even the case.

For some complete theory and applications, see Sections 8.8 and 8.9 of the book [62].

Fourier integral transform case:

As a typical example, we shall examine the Fourier integral transform. For one dimensiona case, we consider the integral transform, for the functions F of $L_2(-\pi t, +\pi t)$, $t > 0$ as

$$f_t(z) = \frac{1}{2\pi} \int_{-\pi t}^{\pi t} F(t) e^{-iz\xi} d\xi. \tag{13.2}$$

In order to identify the image space, following the theory of reproducing kernels, we form the reproducing kernel

$$\begin{aligned} K_t(z, \overline{u}) &= \frac{1}{2\pi} \int_{-\pi t}^{\pi t} e^{-iz\xi} \overline{e^{-iu\xi}} d\xi \qquad (13.3) \\ &= \frac{1}{\pi(z - \overline{u})} \sin \pi t(z - \overline{u}). \end{aligned}$$

The image space of (13.2) is called the Paley-Wiener space $W(\pi t)$ consisting of all the analytic functions of exponential type satisfying, for some constant C and as $z \to \infty$

$$|f_t(z)| \leq C \exp(\pi |z| t)$$

and

$$\int_{\mathbf{R}} |f_t(\xi)|^2 d\xi < \infty.$$

From the identity

$$K_t\left(\frac{j}{t}, \frac{j'}{t}\right) = t\delta(j, j')$$

(the Kronecker's δ), since $\delta(j, j')$ is the reproducing kernel for the Hilbert space ℓ^2, from the general theory of integral transforms and the Parseval's identity we have the isometric identities in (13.2)

$$\frac{1}{2\pi}\int_{-\pi t}^{\pi t} |F(\xi)|^2 d\xi = \frac{1}{t}\sum_{j=-\infty}^{\infty} |f_t(j/t)|^2 = \int_{\mathbf{R}} |f_t(\xi)|^2 d\xi.$$

That is, the reproducing kernel Hilbert space H_{K_t} with $K_t(z, \overline{u})$ is characterized as a space consisting of the Paley-Wiener space $W(\pi t)$ and with the norm squares above. Here we used the well-known result that $\{j/t\}_{j=-\infty}^{\infty}$ is a unique set for the Paley-Wiener space $W(\pi t)$; that is, $f_t(j/t) = 0$ for all j implies $f_t \equiv 0$. Then, the reproducing property of $K_t(z, \overline{u})$ states that

$$f_t(x) = (f_t, K_t(\cdot, x))_{H_{K_t}} = \frac{1}{t}\sum_{j=-\infty}^{\infty} f_t(j/t) K_t(j/t, x) = \int_{\mathbf{R}} f_t(\xi) K_t(\xi, x) d\xi.$$

In particular, on the real line x, this representation is the sampling theorem which represents the whole data $f_t(x)$ in terms of the discrete data $\{f_t(j/t)\}_{j=-\infty}^{\infty}$. For a general theory for the sampling theory and error estimates for some finite points $\{j/t\}_j$, see [55]. As this typical case, we note that all the reproducing kernel Hilbert spaces H_{K_t} may be realized in the space $L^2(\mathbf{R}, d\xi)$ which is now the completion H_∞ of the spaces H_{K_t}.

The representation and approximation by the reproducing kernel $K_t(z, \overline{u})$ are deeply examined and in particular its method is called Sinc method by F. Stenger [65].

14. INVERSION FROM MANY TYPES DATA

We shall give an application to inversion from many kinds of information data.

We suppose that we are given $\mathcal{H}_\lambda$ for each $\lambda \in \Lambda$, where Λ is an abstract set. We are in addition given bounded linear operators

$$L_\lambda \in B(\mathcal{H}, \mathcal{H}_\lambda). \tag{14.1}$$

In particular, with λ fixed, we are interested in the inversion formula

$$L_\lambda x \mapsto x, \quad x \in \mathcal{H}. \tag{14.2}$$

Here, we consider $\{L_\lambda x; \lambda \in \Lambda\}$ as informations obtained from x and we wish to determine x from the informations.

However, the informations $L_\lambda x$ belong to various Hilbert spaces $\mathcal{H}_\lambda$, and so, in order to unify the informations in a sense, we shall take fixed elements $\mathbf{b}_{\lambda,\omega} \in \mathcal{H}_\lambda$ and consider the linear mapping from $\mathcal{H}$

$$X_{\mathbf{b}}(\lambda,\omega) = \langle L_\lambda x, \mathbf{b}_{\lambda,\omega}\rangle_{\mathcal{H}_\lambda} = \langle x, L_\lambda^* \mathbf{b}_{\lambda,\omega}\rangle_{\mathcal{H}}, \quad x \in \mathcal{H} \tag{14.3}$$

into a linear space comprising functions on $\Lambda \times \Omega$. For the informations $L_\lambda x$, we shall consider $X_{\mathbf{b}}(\lambda,\omega)$ as observations (measurements, in fact) for x depending on λ and ω. For this linear mapping (14.3), we form the positive definite quadratic function $K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega')$ on $\Lambda \times \Omega$ defined by:

$$K_{\mathbf{b}}(\lambda,\omega;\lambda',\omega') = \langle L_{\lambda'}^* \mathbf{b}_{\lambda',\omega'}, L_\lambda^* \mathbf{b}_{\lambda,\omega}\rangle_{\mathcal{H}} = \langle L_\lambda L_{\lambda'}^* \mathbf{b}_{\lambda',\omega'}, \mathbf{b}_{\lambda,\omega}\rangle_{\mathcal{H}_\lambda}$$

on $\Lambda \times \Omega$. Then, we can apply our theory. The concept was derived by generalizing the Pythagorean theorem in the following way.

Let $x \in \mathbb{R}^n$ and $\{\mathbf{e}_j\}_{j=1}^n$ be orthogonal unit vectors. We consider the linear mappings

$$L : x \mapsto \{x - \langle x, \mathbf{e}_j\rangle \mathbf{e}_j\}_{j=1}^n \tag{14.4}$$

from $\mathbb{R}^n$ into $\mathbb{R}^n$. Then we wish to establish an isometric identity and inversion formula for the operator. Recall the Pythagorean theorem for $n = 2$. By our operator versions, we can establish the desired results.

Note that in (14.4), for $n \geq 3$ if we instead consider

$$\{\|x - \langle x, \mathbf{e}_j\rangle \mathbf{e}_j\|\}_{j=1}^n \tag{14.5}$$

as scalar valued mappings, then the mappings are no longer linear. So, we must consider the operator valued mappings (14.4) in order to obtain isometric mappings in the framework of Hilbert spaces.

Some related equations were considered as follows [47], 128-157:

Let $H, H_j(E); j = 1, 2, \ldots, p$ be Hilbert spaces and let

$$R_j : H \mapsto H_j(E), \quad j = 1, 2, \ldots, p \tag{14.6}$$

be linear continuous maps from H onto $H_j(E)$. Let $g_j \in H_j(E)$ be given. Then, consider the problem to compute $f \in H$ such that

$$R_j f = g_j, \quad j = 1, 2, \ldots, p. \tag{14.7}$$

These equations are very important in the theory of computerized tomography by the discretization. The typical method is Kaczmarz's method based on an iterative method by using the orthogonal projections P_j in H onto the affine subspaces $R_j f = g_j$.

As for our direct solutions for (14.7) it seems that the result is stable for the sake of (14.7) as data.

Reproducing kernels for nonlinear adaptive filtering tasks have widely been applied and the applications of reproducing kernels for signal analysis are developing; see the references [39, 22, 41, 63, 64, 66]. In particular, for a comprehensive introduction to kernel adaptive filtering, see the book [40].

15. THE AVEIRO DISCRETIZATION METHOD

In general, the reproducing kernel Hilbert space H_K has a complicated structure, and so we have to consider the approximate realization of the abstract Hilbert space H_K by taking a finite number of points of E. A finite number of data will be lead to a discretization principle and practical method, because computers can deal with a finite number of data.

For using a finite number of data, it will be very important in any numerical method, however, in this viewpoint, the theory of reproducing kernels is very good and will give a fairly good nature essentially. Meanwhile, computer power is increasing greatly day after day. And so, some simple algorithm will be required in some general form and for this point, the method stated here will give a general and uniform method.

Finite element methods and difference methods established may be considered very complicated in their natures.

By taking a finite number of points $\{p_j\}_{j=1}^n$, we set

$$K(p_j, p_{j'}) := a_{jj'}. \tag{15.1}$$

Then, if the matrix $A :=\| a_{jj'} \|$ is positive definite, then, the corresponding norm in H_A comprising the vectors $\mathbf{x} = (x_1, x_2, ..., x_n)^T$ is determined by

$$\|\mathbf{x}\|_{H_A}^2 = \mathbf{x}^* \widetilde{A}\mathbf{x},$$

where $\widetilde{A} = \overline{A^{-1}} = \|\widetilde{a_{jj'}}\|$.

When we approximate the reproducing kernel Hilbert space H_K by the vector space H_A, then from Proposition 12.1, the following is directly derived:

Proposition 15.1. *In the linear mapping*

$$f(p) = (\mathbf{f}, \mathbf{h}(p))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{15.2}$$

for

$$\{p_1, p_2, ..., p_n\},$$

the minimum norm inverse $\mathbf{f}_{A_n}^*$ *satisfying*

$$f(p_j) = (\mathbf{f}, \mathbf{h}(p_j))_{\mathcal{H}}, \quad \mathbf{f} \in \mathcal{H} \tag{15.3}$$

is given by

$$\mathbf{f}_{A_n}^* = \sum_{j=1}^n \sum_{j'=1}^n f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \tag{15.4}$$

where $\widetilde{a_{jj'}}$ *are assumed the elements of the complex conjugate inverse of the positive definite Hermitian matrix* A_n *constituted by the elements*

$$a_{jj'} = (\mathbf{h}(p_{j'}), \mathbf{h}(p_j))_{\mathcal{H}} = K(p_j, p_{j'}).$$

Here, the positive definiteness of A_n *is a basic assumption.*

The following proposition shows the convergence of the approximate inverses in Proposition 15.1.

Proposition 15.2. *Let $\{p_j\}_{j=1}^{\infty}$ be a sequence of distinct points on E, that is the positive definiteness for any n and a uniqueness set for the reproducing kernel Hilbert space H_K; that is, for any $f \in H_K$, if all $f(p_j) = 0$, then $f \equiv 0$. Then, in the space $\mathcal{H}$*

$$\lim_{n\to\infty} \mathbf{f}_{A_n}^* = \mathbf{f}^*. \tag{15.5}$$

From the result, we can obtain directly the ultimate realization of the reproducing kernel Hilbert spaces and the ultimate sampling theory:

Proposition 15.3. *(Ultimate realization of reproducing kernel Hilbert spaces). In the general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence in Proposition 15.1,*

$$\|f\|_{H_K}^2 = \|\mathbf{f}^*\|_{\mathcal{H}}^2 = \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\overline{f(p_{j'})}. \tag{15.6}$$

Proposition 15.4. *(Ultimate sampling theory). In the general situation and for a uniqueness set $\{p_j\}$ of the set E satisfying the linearly independence in Proposition 7.1,*

$$f(p) = \lim_{n\to\infty} (\mathbf{f}_{A_n}^*, \mathbf{h}(p))_{\mathcal{H}} = \lim_{n\to\infty} \left(\sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}\mathbf{h}(p_{j'}), \mathbf{h}(p) \right)_{\mathcal{H}} \tag{15.7}$$

$$= \lim_{n\to\infty} \sum_{j=1}^{n} \sum_{j'=1}^{n} f(p_j)\widetilde{a_{jj'}}K(p, p_{j'}).$$

In Proposition 15.1, for any given finite number $f(p_j)$, $j = 1, 2, ..., n$, the result in Proposition 15.1 is valid. Meanwhile, Proposition 15.2 and Proposition 15.4 are valid when we consider the sequence $f(p_j)$, $j = 1, 2, ...$, for any member f of H_K. The sequence $f(p_j)$, $j = 1, 2, ...$, for any member f of H_K is characterized by the convergence of (15.6) in Proposition 15.3. Then, any member f of H_K is represented by (15.7) in terms of the sequence $f(p_j)$, $j = 1, 2, ...$, in Proposition 15.4.

In the general setting in Proposition 15.1, assume that we observed the values $f(p_j) = \alpha_j, j = 1, 2, ..., n$, for a finite number of points $\{p_j\}$, then for the whole value $f(p)$ of the set E, how can we consider it?

One idea is to consider the function $f_1(p)$: among the functions satisfying the conditions $f(p_j) = \alpha_j, j = 1, 2, ..., n$, we consider the minimum norm member $f_1(p)$ in $H_K(E)$. This function $f_1(p)$ is determined by the formula,

$$f_1(p) = \sum_{j=1}^{n} C_j K(p, p_j)$$

where, the constants $\{C_j\}$ are determined by the formula:

$$\sum_{j=1}^{n} C_j K(p_{j'}, p_j) = \alpha_{j'}, j' = 1, 2, ..., n.$$

(of course, we assume that $\|K(p_{j'}, p_j)\|$ is positive definite).

For this problem, see, Mo, Y. and Qian, T. : Support vector machine adapted Tikhonov regularization method to solve Dirichlet problem ([48]), as a new numerical approarch by a usual computer system level. In particular, they can deal with errorness data. We use a special powerful computer system by H. Fujiwara based on the infinite precision method and parallel computer systems. Indeed, Fujiwara can consider over 600 digits and use over 1000 computers at the same time.

Meanwhile, by Proposition 15.4, we can consider the function $f_2(p)$ defined by

$$f_2(p) = (\mathbf{f}^*_{A_n}, \mathbf{h}(p))_{\mathcal{H}}$$

in terms of $\mathbf{f}^*_{A_n}$ in Proposition 15.1. This interpolation formula is depending on the linear system.

For analytical problems, we need discretization and using a finite number of data in order to obtain approximate solutions by using computers, the typical methods are finite element method and difference method, however, their practical algorithms will be complicated depending on case by case, depending on the domains and depending on the dimensions, however, the above methods are essentially simple and uniform method in principle, called the *Aveiro discretization method*. However, the hard work part is to solve the linear simultaneous equations, computer powers requested are increasing and so, in future, the above simple method may be expected to become a standard method.

Many numerical experiments for the sampling theory by Proposition 15.4 were given by [29].

We showed a general sampling theorem and the concrete numerical experiments for the simplest and typical examples. We gave the sampling theorem in the Sobolev Hilbert spaces with numerical experimences. For the Sobolev Hilbert spaces, sampling theorems seem to be a new concept.

For the typical Paley-Wiener spaces, the sampling points are automatically determined as the common sense, however, in our general sampling theorem, we can select the sampling points freely and so, case by case, following some *a priori* information of a considering function, we can take the effective sampling points. We showed these properties by the concrete examples, by many Figures.

A Typical Example of the Aveiro Discretization Method With ODE:

Consider a prototype differential operator

$$Ly := \alpha y'' + \beta y' + \gamma y. \tag{15.8}$$

Here, consider a very general situation that the coefficients are *arbitrary* functions on their nature and on a general interval I.

For some practical construction of some natural solution of

$$Ly = g \tag{15.9}$$

for a very general function g on a general interval I,

Proposition 15.5. *Let us fix a positive number h and take a finite number of points $\{t_j\}_{j=1}^n$ of I such that*

$$(\alpha(t_j), \beta(t_j), \gamma(t_j)) \neq \mathbf{0}$$

for each j. Then, an optimal solution y_h^A of the equation (15.9) is given by

$$y_h^A(t) = \frac{1}{2\pi} \int_{-\pi/h}^{\pi/h} F_h^A(\xi) e^{-it\xi} d\xi$$

in terms of the function $F_h^A \in L_2(-\pi/h, +\pi/h)$ in the sense that F_h^A has the minimum norm in $L_2(-\pi/h, +\pi/h)$ among the functions $F \in L_2(-\pi/h, +\pi/h)$ satisfying, for the characteristic function $\chi_h(t)$ of the interval $(-\pi/h, +\pi/h)$:

$$\frac{1}{2\pi} \int_{\mathbf{R}} F(\xi)[\alpha(t)(-\xi^2) + \beta(t)(-i\xi) + \gamma(t)]\chi_h(\xi) \exp(-it\xi) d\xi = g(t) \tag{15.10}$$

for all $t = t_j$ and for the function space $L_2(-\pi/h, +\pi/h)$.

The best extremal function F_h^A is given by

$$F_h^A(\xi) = \sum_{j,j'=1}^n g(t_j)\widetilde{a_{jj'}}\overline{(\alpha(t_{j'})(-\xi^2) + \beta(t_{j'})(-i\xi) + \gamma(t_{j'}))} \exp(it_{j'}\xi). \tag{15.11}$$

Here, the matrix $A = \{a_{jj'}\}_{j,j'=1}^n$ formed by the elements

$$a_{jj'} = K_{hh}(t_j, t_{j'})$$

with

$$\begin{aligned} K_{hh}(t,t') &= \frac{1}{2\pi} \int_{\mathbf{R}} [\alpha(t)(-\xi^2) + \beta(t)(-i\xi) + \gamma(t)]\overline{[\alpha(t')(-\xi^2) + \beta(t')(-i\xi) + \gamma(t')]} \\ &\quad \cdot \chi_h(\xi) \exp(-i(t-t')\xi) d\xi \end{aligned} \tag{15.12}$$

is positive definite and the $\widetilde{a_{jj'}}$ are the elements of the inverse of $\overline{A}$ (the complex conjugate of A).

Therefore, the optimal solution y_h^A of the equation (15.9) is given by

$$\begin{aligned} y_h^A(t) = \sum_{j,j'=1}^n g(t_j)\widetilde{a_{jj'}} \frac{1}{2\pi} [-\overline{\alpha(t_{j'})} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi^2 e^{-i(t-t_{j'})\xi} d\xi \\ + i\overline{\beta(t_{j'})} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} \xi e^{-i(t-t_{j'})\xi} d\xi + \overline{\gamma(t_{j'})} \int_{-\frac{\pi}{h}}^{\frac{\pi}{h}} e^{-i(t-t_{j'})\xi} d\xi]. \end{aligned}$$

At first, we are considering approximate solutions of the differential equation (15.9) and at this point, we are considering the Paley-Wiener function spaces with parameter h as approximating function spaces. Next, by using the Fourier inversion, the differential equation (15.9) may be transformed to (15.10). However, to solve the integral equation (15.10) is very difficult for the generality of the coefficient functions. So, we assume (15.10) is valid on some finite number of points t_j. This assumption will be very reasonable for the discretization of the integral equation. By this assumption we can obtain an optimal approximate solutions in a very simple way.

Here, we assume that equation (15.9) is valid on I and so, as some practical case we would like to consider the equation in (15.9) on I satisfying some boundary conditions. In the present case, the boundary conditions are given as zero at infinity for $I = \mathbf{R}$.

However, our result gives the approximate general solutions satisfying boundary values. For example, for a finite interval (a, b), we consider $t_1 = a$ and $t_n = b$ and $\alpha(t_1) = \beta(t_1) = \alpha(t_n) = \beta(t_n) = 0$. Then, we can obtain the approximate solution having the arbitrary given boundary values $y_h^A(t_1)$ and $y_h^A(t_n)$. In addition, by a simple modification we may give the general approximate solutions satisfying the corresponding boundary values.

For a finite interval case I, following the boundary conditions, we can consider the corresponding reproducing kernels by the Sobolev Hilbert spaces. However, the concrete representations of the reproducing kernels are involved depending on the boundary conditions. However, we can still consider them and we can use them.

Of course, for a smaller h we can obtain a better approximate solution.

For the representation (15.12) of the reproducing kernel $K_{hh}(t, t')$, we can calculate it easily.

The very surprising facts are: for variable coefficients linear differential equations, we can represent their approximate solutions satisfying their boundary conditions without integrals. Approximate function spaces may be considered with the Paley-Wiener spaces and the Sobolev spaces. For many concrete examples and numerical examples, see [15, 16]. We showed Figures of the numerical experiments. See also [51] for some applications to nonlinear partial differential equations.

16. REPRESENTATION OF INVERSE FUNCTIONS

By using the theory of reproducing kernels, **for an arbitrary mapping** we will be able to consider some representation of its inversion. For example, we can consider the following concrete problems:

Let $\varphi : E' \to E$ be a bijection. Then,

$$f \circ \varphi(p) = f(\varphi(p)) = \langle f, K_{\varphi(p)} \rangle_{H_K(E)}, \tag{16.1}$$

and we obtain

$$\varphi^{-1}(p) = \langle \varphi^{-1}, K_p \rangle_{H_K(E)} = \langle \mathrm{id}, K(\varphi(\cdot), p) \rangle_{H_{\varphi^* K}(E')} \tag{16.2}$$

if $\varphi^{-1} \in H_K(E)$.

Precisely, see [62], Section 8.3.2. However, its idea is indeed very simple. As in (16.2), we represent the inverse function by some reproducing kernel and next, we transform the representation by the mapping.

The following simple result was derived by representing the inverse of the Riemann mapping function on the unit disk in terms of the Bergman kernel and by the transform by the Riemann mapping function, however, the result may be derived directly:

Example 16.1. Suppose that $\varphi : \Omega_1 \to \Delta(1)$ is a biholomorphic function. Then we have

$$\varphi^{-1}(z) = \frac{1}{\pi}\int_{\Omega_1} \frac{z|\varphi'(z)|^2}{(1-z\overline{\varphi(z)})^2}\,dx\,dy \tag{16.3}$$

for all $z \in \Delta(1)$.

Example 16.2. Let $K(x,y) \equiv \min(x,y)$ for $0 \le x, y < \infty$. Then we have

$$H_K[0,\infty) = \left\{f \in W^{1,2}[0,\infty) \,:\, f(0) = 0\right\}. \tag{16.4}$$

Let $\varphi : [0,\infty) \to [0,\infty)$ be an increasing function such that $\varphi|(0,\infty) \in C^1(0,\infty)$ that $\varphi(0) = 0$, and that $\varphi'(x) > 0$ for almost every $x > 0$. First observe that

$$\varphi^*K(x,y) = K(\varphi(x),\varphi(y)) = \varphi(K(x,y)). \tag{16.5}$$

Thus, we have

$$H_{\varphi^*K} = \{f \in AC[0,\infty) \,:\, \|f\|_{H_{\varphi^*K}} < \infty\}$$

and

$$\|f\|_{H_{\varphi^*K}} = \sqrt{\int_0^\infty f'(\xi)^2 \frac{d\xi}{\varphi'(\xi)}} < \infty. \tag{16.6}$$

Observe also that $\displaystyle\int_0^\infty \frac{\sin at}{t}\,dt = \frac{\pi}{2}\mathrm{sgn}(a)$, which is known as the Dirichlet integral. Note that

$$\begin{aligned}\varphi^{-1}(x) = \langle \mathrm{id}, \varphi^*K(\cdot,\varphi^{-1}(x))\rangle_{H_{\varphi^*K}} &= \langle \mathrm{id}, K(\varphi(\cdot),\varphi(\varphi^{-1}(x)))\rangle_{H_{\varphi^*K}} \\ &= \langle \mathrm{id}, K(\varphi(\cdot),x)\rangle_{H_{\varphi^*K}}.\end{aligned}$$

As a result, we obtain

$$\begin{aligned}\varphi^{-1}(x) &= \int_0^\infty \frac{d}{d\xi}\min(\varphi(\xi),x)\,\frac{d\xi}{\varphi'(\xi)} \\ &= \frac{2}{\pi}\int_0^\infty \left(\varphi'(\xi)\int_0^\infty \frac{\cos(\varphi(\xi)t)\sin x\,t}{t}\,dt\right)\frac{d\xi}{\varphi'(\xi)} \\ &= \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty \frac{\cos(\varphi(\xi)t)\sin x\,t}{t}\,dt\right)d\xi.\end{aligned}$$

In particular, by letting $\varphi(x) = x^n$,

$$\sqrt[n]{x} = \frac{2}{\pi}\int_0^\infty \left(\int_0^\infty \frac{\cos(\xi^n\,t)\sin x\,t}{t}\,dt\right)d\xi \tag{16.7}$$

for all $n \in \mathbb{N}$ and $x > 0$.

Example 16.3. Suppose that $f : [a,b] \to \mathbb{R}$ is an increasing function such that $f' \in C[a,b]$. Then as is well known, f^{-1} belongs to the Sobolev class $W^{1,2}[a,b]$, whose inner product is given by:

$$\langle f_1, f_2\rangle_{W^{1,2}[a,b]} = f_1(a)f_2(a) + f_1(b)f_2(b) + \int_a^b \left(f_1(x)f_2(x) + f_1'(x)f_2'(x)\right)dx. \tag{16.8}$$

Note that the kernel of this RHKS is given by: $K(x,y) = \frac{1}{2}\exp(-|x-y|)$. Then, by the formal way, we obtain

$$f^{-1}(y_0) = \frac{a+b}{2} + \frac{1}{2}\int_a^b \operatorname{sgn}(y_0 - f(x))\, dx. \tag{16.9}$$

This formula (16.9) can be, however, derived directly and simply, and we note that we do not need any smoothness assumptions on the function f. Indeed, we need only the strictly increasing assumption.

Denote by $\alpha(n)$ the volume of the unit ball in $\mathbb{R}^n$. We can naturally generalize the above observations to $\mathbb{R}^n$ and this may be considered as a counterpart to non-linear simultaneous equations of Kramer's formula for regular matrices:

Proposition 16.4. *Let $n \geq 3$. Let Ω be a bounded C^1-domain in $\mathbb{R}^n$ and let $f \in C^1(\overline{\Omega})$, that is, f is defined on an open set U containing $\overline{\Omega}$. Assume further that $f : U \mapsto f(U)$ is an orientation preserving mapping. Then, for $y_0 \in f(D)$, we have*

$$f_i^{-1}(y_0) = -\int_\Omega dx_i \wedge f^*[*dG_n(\cdot - y_0)] + \int_{\partial\Omega} x_i f^*[*dG_n(\cdot - y_0)]. \tag{16.10}$$

Here, f_i^{-1} denotes the i component of f^{-1}, $*$ denotes the Hodge star operator, G_n the fundamental solution of the Laplacian $\Delta = \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$:

$$G_n(z) = \frac{1}{n(n-2)\alpha(n)|z|^{n-2}}.$$

In particular, for $n = 1$, we obtain (16.9), directly.

For $n = 2$, we obtain the following:

Proposition 16.5. *Let Ω be a bounded C^1-domain in $\mathbb{R}^2$ and let $f \in C^1(\overline{\Omega})$, that is, f is defined on an open set U containing $\overline{\Omega}$. Assume further that $f : U \mapsto f(U)$ is an orientation preserving mapping. For any $\hat{y} = (\hat{y}_1, \hat{y}_2) \in f(D)$, we have*

$$2\pi f_1^{-1}(\hat{y}) = \int_{\partial D} x_1 d\left[\operatorname{Arc}\tan\left(\frac{f_1(x) - \hat{y}_1}{f_2(x) - \hat{y}_2}\right)\right] - \int_D dx_1 \wedge d\left[\operatorname{Arc}\tan\left(\frac{f_1(x) - \hat{y}_1}{f_2(x) - \hat{y}_2}\right)\right]$$

$$2\pi f_1^{-1}(\hat{y}) = \int_D dx_2 \wedge d\left[\operatorname{Arc}\tan\left(\frac{f_2(x) - \hat{y}_2}{f_1(x) - \hat{y}_1}\right)\right] - \int_{\partial D} x_2 d\left[\operatorname{Arc}\tan\left(\frac{f_2(x) - \hat{y}_2}{f_1(x) - \hat{y}_1}\right)\right].$$

Note that the differential forms

$$d\left[\operatorname{Arc}\tan\left(\frac{f_1(x) - \hat{y}_1}{f_2(x) - \hat{y}_2}\right)\right] \text{ and } d\left[\operatorname{Arc}\tan\left(\frac{f_2(x) - \hat{y}_2}{f_1(x) - \hat{y}_1}\right)\right]$$

make sense despite the ambiguity of the choice of the branch of Arc tan.

Proposition 16.5 is represented more explicitly as follows: Let $\Omega \subset \mathbb{R}^2$ be a bounded domain with a finite number of piecewise C^1 class boundary components.

Let f be a one-to-one C^1 class mapping from $\overline{\Omega}$ into $\mathbb{R}^2$ and we assume that its Jacobian $J(x)$ is positive on Ω. We shall represent f as follows:

$$\begin{cases} y_1 = f_1(x) = f_1(x_1, x_2) \\ y_2 = f_2(x) = f_2(x_1, x_2) \end{cases} \tag{16.11}$$

and the inverse mapping f^{-1} of f as follows:

$$\begin{cases} x_1 = (f^{-1})_1(y) = (f^{-1})_1(y_1, y_2) \\ x_2 = (f^{-1})_2(y) = (f^{-1})_2(y_1, y_2). \end{cases} \tag{16.12}$$

Proposition 16.6. *For the mappings* (16.11) *and* (16.12)*, we have*

$$\begin{pmatrix} 2\pi (f^{-1})_1(y^*) \\ 2\pi (f^{-1})_2(y^*) \end{pmatrix}$$

$$= \oint_{\partial D} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} d\left[\operatorname{Arc}\tan\left(\frac{f_2(x) - y_2^*}{f_1(x) - y_1^*}\right)\right] - \int\int_D \frac{\operatorname{adj} J(x)}{|f(x) - y^*|^2} \begin{pmatrix} f_1(x) - y_1^* \\ f_2(x) - y_2^* \end{pmatrix} dx_1 dx_2.$$

for any $y^* = (y_1^*, y_2^*) \in f(\Omega)$.

The fundamental application of Proposition 16.6 is the identification of the solution space; because for the outer side of the solutions, the representations in the right-hand side in Proposition 16.6 are zero and applications to the implicit function theory, because we can represent the explicit functions explicitly whose existence is guaranteed by the implicit function theory. For the proof of Proposition 16.6 and its application to the implicit function theory, see the next with [62], Section A.3.

17. REPRESENTATIONS OF IMPLICIT FUNCTIONS

We shall introduce the fundamental result that is obtained as a natural extension.

Proposition 17.1. *Let* $U \subset \mathbb{R}^{n+k}$ *be a smooth bounded domain surrounded by a finite number of* C^1*–class and simple closed surfaces. For* k *functions*

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}), \quad i = 1, 2, \ldots, k, \tag{17.1}$$

we assume that for some point on U *it holds*

$$f_i(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = 0, \quad i = 1, 2, \ldots, k, \tag{17.2}$$

and that, on U*, we have*

$$\det \frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x) > 0. \tag{17.3}$$

In this way, for each $j = 1, 2, \ldots, k$*, we also assume globally that* C^1*-functions* $g_j(x_1, x_2, \ldots, x_n)$ *on* $U \cap \mathbb{R}^n$ *satisfies*

$$f_i(x_1, x_2, \ldots, x_n, g_1, g_2, \ldots, g_k) = 0, \quad i = 1, 2, \ldots, k, \tag{17.4}$$

and

$$x_{n+j} = g_j(x_1, x_2, \ldots, x_n), \quad j = 1, 2, \ldots, k. \tag{17.5}$$

Then, for $j = 1, 2, \ldots, k$, *it holds*

$$\begin{aligned} & g_j(x_1, \ldots, x_n) \\ = & \sum_{i=1}^{n+k} \frac{(-1)^{n+j+i-1}}{c_{n+k} A_{n+k}} \int_U \frac{(\eta - \eta_0)_i}{|\eta - \eta_0|^{n+k}} \det \frac{\partial(\eta_1, \ldots, \eta_{i-1}, \eta_{i+1}, \ldots, \eta_{n+k})}{\partial(\xi_1, \ldots, \xi_{n+j-1}, \xi_{n+j+1}, \ldots, \xi_{n+k})}(\xi)\, d\xi \\ & + \int_{\partial U} \xi_{n+j} F^* * dG_{n+k}(\eta - \eta_0), \end{aligned}$$

where $A_n = 2\pi^{\frac{n}{2}}/\Gamma(\frac{n}{2})$ *is the surface measure of the* n *dimensional unit disk and* F *is a mapping from* U *into* $\mathbb{R}^{n+k}$ *such that:*

$$F(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \\ f_1(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \\ f_2(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \\ \vdots \\ f_k(x_1, \ldots, x_n, x_{n+1}, \ldots, x_{n+k}) \end{pmatrix}. \tag{17.6}$$

Proof. In order to apply Proposition 16.4, first we shall fix the direct mapping in Proposition 16.4 for our situation.

We will consider the C^1–class mapping F from U into $\mathbb{R}^{n+k}$ introduced in (17.6). It is clear that the Jacobian of this mapping is not vanishing on U as in

$$\det F'(x_1, \ldots, x_{n+k}) = \det \frac{\partial(f_1, f_2, \ldots, f_k)}{\partial(x_{n+1}, x_{n+2}, \ldots, x_{n+k})}(x) > 0. \tag{17.7}$$

By assumption, since the mapping F is injective on U, we can consider its inversion on its image domain. In particular, for any $(x_1, \ldots, x_n, 0, \ldots, 0)$ of the image domain that is the restriction to the domain U, we have from the situation in Section 16, on $U \cap \mathbb{R}^n$,

$$\begin{aligned} & F|_U^{-1}(x_1, x_2, \ldots, x_n, 0, \ldots, 0) \\ = \; & F|_U^{-1}\Big(x_1, \ldots, x_n, f_1\big(x_1, \ldots, x_n, g_1, \ldots, g_k\big), \ldots, f_k\big(x_1, \ldots, x_n, g_1, \ldots, g_k\big)\Big) \\ = \; & \begin{pmatrix} x_1 \\ \vdots \\ x_n \\ g_1 \\ \vdots \\ g_k \end{pmatrix}. \end{aligned} \tag{17.8}$$

Therefore, by the representation in Proposition 16.4, we obtain the identities for the explicit functions g_i,

$$g_i(x_1, \ldots, x_n) = -\int_U d\xi_{n+i} \wedge F^* * dG_{n+k}(\eta - \eta_0) + \int_{\partial U} \xi_{n+i} F^* * dG_{n+k}(\eta - \eta_0), \tag{17.9}$$

for $(x_1, \ldots, x_n) \in U \cap \mathbb{R}^n$. Here,

$$\xi = (\xi_1, \ldots, \xi_n, \xi_{n+1}, \ldots, \xi_{n+i}, \ldots, \xi_{n+k}), \quad (i = 1, 2, \ldots, k), \tag{17.10}$$

and

$$\eta - \eta_0 = (\xi_1 - x_1, \ldots, \xi_n - x_n, f_1(\xi_1, \ldots, \xi_{n+k}), \ldots, f_k(\xi_1, \ldots, \xi_{n+k})). \tag{17.11}$$

Recalling that A_n is the surface measure of the n dimensional unit disk, we have, precisely

$$G_n(x) = \frac{1}{c_n A_n} \begin{cases} |x|, & n = 1 \\ \log |x|, & n = 2 \ \text{(logarithmic kernel)} \\ -|x|^{-n+2}, & n \geq 3 \ \text{(Newton kernel).} \end{cases}$$

Hence, on $\mathbb{R}^n \setminus U_\varepsilon(0)$ we have

$$dG_n(x) = \frac{1}{c_n A_n |x|^n} \sum_{i=1}^{n} x_i dx_i.$$

Therefore, by definition,

$$*dG_n(x) = \frac{1}{c_n A_n |x|^n} \sum_{i=1}^{n} (-1)^{i-1} x_i dx_1 \wedge \ldots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \ldots \wedge dx_n, \tag{17.12}$$

for $x = (x_1, \ldots, x_n)$.

As a consequence, it holds

$$\begin{aligned} &*dG_{n+k}(\eta - \eta_0) \\ &= \sum_{i=1}^{n+k} \frac{(-1)^{i-1}(\eta - \eta_0)_i}{c_{n+k} A_{n+k} |\eta - \eta_0|^{n+k}} d\eta_1 \wedge \ldots \wedge d\eta_{i-1} \wedge d\eta_{i+1} \wedge \ldots \wedge d\eta_{n+k}. \end{aligned}$$

Then, the pull back $F^* * dG_{n+k}(\eta - \eta_0)$ needed in the representation of the explicit functions is computed by the following general formula and the Jacobian

$$y^*(dy_1 \wedge \ldots \wedge dy_{j-1} \wedge dy_{j+1} \wedge \ldots \wedge dy_n)$$

$$= \sum_{k=1}^{n} \det \left(\frac{\partial(y_1, \ldots, y_{j-1}, y_{j+1}, \ldots, y_n)}{\partial(x_1, \ldots, x_{k-1}, x_{k+1}, \ldots, x_n)}(x) \right) dx_1 \wedge \ldots \wedge dx_{k-1} \wedge dx_{k+1} \wedge \ldots \wedge dx_n.$$

Indeed,

$$
\begin{aligned}
&F^* * dG_{n+k}(\eta-\eta_0)\\
&= \frac{1}{c_{n+k}A_{n+k}|\eta-\eta_0|^{n+k}}\\
&\quad \cdot \sum_{i=1}^{n+k}(-1)^{i-1}(\eta-\eta_0)_i \sum_{p=1}^{n+k} \det\left(\frac{\partial(\eta_1,\ldots,\eta_{i-1},\eta_{i+1},\ldots,\eta_{n+k})}{\partial(\xi_1,\ldots,\xi_{p-1},\xi_{p+1},\ldots,\xi_{n+k})}(\xi)\right)\\
&\quad \cdot d\xi_1\wedge\ldots\wedge d\xi_{p-1}\wedge d\xi_{p+1}\wedge\ldots\wedge d\xi_{n+k}\\
&= \frac{1}{c_{n+k}A_{n+k}|\eta-\eta_0|^{n+k}}\\
&\quad \cdot \sum_{p=1}^{n+k}\Big(\sum_{i=1}^{n+k}(-1)^{i-1}(\eta-\eta_0)_i \det\frac{\partial(\eta_1,\ldots,\eta_{i-1},\eta_{i+1},\ldots,\eta_{n+k})}{\partial(\xi_1,\ldots,\xi_{p-1},\xi_{p+1},\ldots,\xi_{n+k})}(\xi)\Big)\\
&\quad \cdot \xi_1\wedge\ldots\wedge d\xi_{p-1}\wedge d\xi_{p+1}\wedge\ldots\wedge d\xi_{n+k}.
\end{aligned}
$$

Therefore, the desired representation of g_j (for $j=1,2,\ldots,k$) is obtained. □

17.1. The two–dimensional case: $n=1, k=1$. We shall state the concrete representation formula for the two–dimensional case. From the expression

$$
\begin{aligned}
F^* * dG_2(\eta-\eta_0) &= \frac{1}{c_2A_2((\xi_1-x_1)^2+f_1(\xi_1,\xi_2)^2)}\\
&\quad \cdot\left\{(\xi_1-x_1)\frac{\partial f_1}{\partial\xi_1}d\xi_1+(\xi_1-x_1)\frac{\partial f_1}{\partial\xi_2}d\xi_2-f_1(\xi_1,\xi_2)d\xi_1\right\},
\end{aligned}
$$

we obtain

Proposition 17.2. *For a C^1–class function $f(x_1,x_2)$ on a domain U in $\mathbb{R}^2$, we assume that for a point $x^0=\begin{pmatrix}x_1^0\\ x_2^0\end{pmatrix}$:*

$$\frac{\partial f}{\partial x_2}(x_1^0,x_2^0)\neq f(x_1^0,x_2^0)=0.$$

Then:

(1) *There exist a neighbourhood $U_1\times U_2$ $(\subset U)$ around the point x^0 and an explicit function $g: U_1\to U_2$ determined by the implicit function $f=0$ as $f(x_1,g(x_1))=0$ and, furthermore, it is represented as follows:*

$$g(x_1^*)=\frac{1}{2\pi}\left\{\int_{\partial(U_1\times U_2)}x_2d\theta-\int_{U_1\times U_2}dx_2\wedge d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1-x_1^*}\right)\right]\right\},$$

for any $x_1^*\in U_1$.

(2) *For any $x_1^*\in U_1$, it holds*

$$2\pi x_1^* = \int_{\partial(U_1\times U_2)}x_1d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1-x_1^*}\right)\right]-\int_{U_1\times U_2}dx_1\wedge d\left[\mathrm{Arctan}\left(\frac{f(x_1,x_2)}{x_1-x_1^*}\right)\right].$$

Corollary 17.3 (Representations of exact differential equations)**.** *Let $f(x,y)$ be the C^1–class solution of the differential equation $f_x dx + f_y dy = 0$ on some domain $I_x \times I_y$ on $\mathbb{R}^2$ satisfying $\frac{\partial f(x,y)}{\partial y} \neq 0$ and $y(x_0) = y_0, (x_0, y_0) \in I_x \times I_y$. Then, we obtain the representation of the explicit function $y = y(x)$ which is determined by the implicit function $f(x,y) - f(x_0, y_0) = 0$ for any $x^* \in I_x$,*

$$\begin{aligned} y(x^*) &= \frac{1}{2\pi} \int_{\partial(I_1 \times I_2)} yd \left[\text{Arctan} \left(\frac{f(x,y) - f(x_0,y_0)}{x - x^*} \right) \right] \\ &- \frac{1}{2\pi} \int_{I_1 \times I_2} dy \wedge d \left[\text{Arctan} \left(\frac{f(x,y) - f(x_0,y_0)}{x - x^*} \right) \right]. \end{aligned}$$

Corollary 17.4 (Representations of the inverse functions)**.** *On an open interval $[a,b]$, for a C^1–class function f satisfying $f'(x) > 0$, its inverse function $f^{-1}(y^*)$ on $[f(a), f(b)]$ can be represented as follows:*

$$\begin{aligned} f^{-1}(y^*) &= \frac{1}{2\pi} \int_{[a,b]\times[f(a),f(b)]} dx \wedge d \left[\text{Arctan} \left(\frac{y - f(x)}{y - y^*} \right) \right] \\ &\quad - \frac{1}{2\pi} \int_{\partial([a,b]\times[f(a),f(b)])} xd \left[\text{Arctan} \left(\frac{y - f(x)}{y - y^*} \right) \right] \end{aligned}$$

for any $y^* \in [f(a), f(b)]$.

18. BEST APPROXIMATIONS

Let L be any bounded linear operator from a reproducing kernel Hilbert space H_K into a Hilbert space $\mathcal{H}$. Then, the following problem is a classical and fundamental problem known as the best approximate mean square norm problem: For any member $\mathbf{d}$ of $\mathcal{H}$, we would like to find

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}}.$$

It is clear that we are considering *operator equations*, generalized solutions and corresponding generalized inverses within the framework of $f \in H_K$ and $\mathbf{d} \in \mathcal{H}$, having in mind

$$Lf = \mathbf{d}. \tag{18.1}$$

However, this problem has a complicated structure, specially in the infinite dimension Hilbert spaces case, leading in fact to the consideration of generalized inverses (in the Moore-Penrose sense). Following the reproducing kernel theory, we can realize its complicated structure. Anyway, the problem turns to be well-posed within the reproducing kernels theory framework in the following proposition:

Proposition 18.1. *For any member $\mathbf{d}$ of $\mathcal{H}$, there exists a function $\tilde{f}$ in H_K satisfying*

$$\inf_{f \in H_K} \|Lf - \mathbf{d}\|_{\mathcal{H}} = \|\mathbf{L}\tilde{\mathbf{f}} - \mathbf{d}\|_{\mathcal{H}} \tag{18.2}$$

if and only if, for the reproducing kernel Hilbert space H_k admitting the kernel defined by $k(p,q) = (L^* LK(\cdot, q), L^* LK(\cdot, p))_{H_K}$

$$L^*\mathbf{d} \in H_k. \tag{18.3}$$

Furthermore, when there exists a function $\tilde{f}$ satisfying (18.2), there exists a uniquely determined function that minimizes the norms in H_K among the functions satisfying the equality, and its function $f_{\mathbf{d}}$ is represented as follows:

$$f_{\mathbf{d}}(p) = (L^*\mathbf{d}, L^*LK(\cdot, p))_{H_k} \quad on \quad E. \tag{18.4}$$

Here, the adjoint operator L^* of L, as we see, from

$$(L^*\mathbf{d})(p) = (L^*\mathbf{d}, K(\cdot, p))_{H_K} = (\mathbf{d}, LK(\cdot, p))_{\mathcal{H}}$$

is represented by known $\mathbf{d}, L, K(p,q)$, and $\mathcal{H}$. From this Proposition 18.1, we see that the problem is well-established by the theory of reproducing kernels, that is the existence, uniqueness and representation of the solutions in the problem are well-formulated. In particular, note that the adjoint operator is represented in a good way; this fact will be very important. The extremal function $f_{\mathbf{d}}$ is the *Moore-Penrose generalized inverse* $L^{\dagger}\mathbf{d}$ of the equation $Lf = \mathbf{d}$. The criteria (18.3) is involved and so the Moore-Penrose generalized inverse $f_{\mathbf{d}}$ is not good, when the data contain error or noises in some practical cases.

We will refer to the simple and typical applications to best approximation problems with a concrete example.

Let E be an arbitrary set, and let $H_K(E)$ be a RKHS admitting the reproducing kernel $K(p,q)$. Meanwhile, for any subset X of E we consider a Hilbert space $H(X)$ comprising functions F on X. In the relationship of two Hilbert spaces $H_K(E)$ and $H(X)$, we assume the following;

(1) for the restriction $f|_X$ of the members f of $H_K(E)$ to the set X, $f|_X$ belongs to the Hilbert space $H(X)$, and
(2) the restriction operator $Lf = f|X$ is continuous from $H_K(E)$ into $H(X)$.

Then, we consider the fundamental problem

$$\inf_{f \in H_K(E)} \|Lf - F\|_{H(X)} \tag{18.5}$$

for a member F of $H(X)$.

For the sake of the good properties of L and its adjoint L^* in our situation, we can obtain *'algorithms'* to decide the best one f^* of F in the sense of

$$\inf_{f \in H_K(E)} \|Lf - F\|_{H(X)} = \|Lf^* - F\|_{H(X)}, \tag{18.6}$$

when there exists. Furthermore, when there exist best approximations of f^*, we can obtain the best one f^* in a reasonable and constructive way. Indeed, we can obtain intrinsic representations of the best approximation in terms of F and the reproducing kernel $K(p,q)$.

As a typical example, we examine best approximations of functions on the real line by entire functions. Since we need a concrete form of the reproducing kernel as a typical reproducing kernel Hilbert space, for entire functions, we consider the Fischer space $\mathcal{F}_a(\mathbb{R})$ on $\mathbb{C}$ normed by

$$\|f\|_{\mathcal{F}_a(\mathbb{R})} = a\sqrt{\frac{1}{\pi}\iint_{\mathbb{C}} |f(z)|^2 \exp(-a^2|z|^2)\, dx\, dy} \tag{18.7}$$

for fixed $a > 0$, whose reproducing kernel is

$$K_a(z, \overline{u}) = \exp(a^2 \overline{u} z) \quad (z, u \in \mathbb{C}). \tag{18.8}$$

Meanwhile, as a function space approximated by the Fischer space $\mathcal{F}_a(\mathbb{R})$ we shall first determine an $L^2(\mathbb{R}, W(x)dx)$ space with a natural weight $W(x)(\geq 0)$

$$\|F\|_{L^2(W)} = \sqrt{\int_{\mathbb{R}} |F(x)|^2 W(x)\, dx} \tag{18.9}$$

in connection with the Fischer space $\mathcal{F}_a(\mathbb{R})$.

Under these situations we examine the best approximation problem

$$\inf_{f \in F_a} \|Tf - F\|_{L^2(W)} \tag{18.10}$$

for $F \in L^2(W)$.

In this case, the set $\{Tf; f \in \mathcal{F}_a(\mathbb{R})\}$ will be complete in $L^2(W)$ and so we have

$$\inf_{f \in \mathcal{F}_a(\mathbb{R})} \|Tf - F\|_{L^2(W)} = 0. \tag{18.11}$$

Therefore, the condition for the existence of the best approximation f^* in the sense

$$\|Tf^* - F\|_{L^2(W)} = 0 \tag{18.12}$$

will become the condition that F can be extended analytically to the member $f^* \in \mathcal{F}_a(\mathbb{R})$ except for a null Lebesgue measure set on the real line $\mathbb{R}$.

Furthermore, we can construct a sequence $\{f_n\}_{n=0}^{\infty} \subset \mathcal{F}_a(\mathbb{R})$ such that

$$\lim_{n \to \infty} \|Tf_n - F\|_{L^2(W)} = 0 \tag{18.13}$$

for any function F in $L^2(W)$.

We first look for a natural weight W such that the restriction operator T is bounded from $\mathcal{F}_a(\mathbb{R})$ into $L^2(W)$. Note that for any member $f \in \mathcal{F}_a(\mathbb{R})$, the integrals exist by Fubini's theorem

$$\int_{-\infty}^{\infty} |f(x + iy)|^2 \exp(-a^2(x^2 + y^2))\, dx \tag{18.14}$$

for almost all $y \in \mathbb{R}$. In this setting, we have the following inequality (when $y = 0$):

Proposition 18.2. *Let $\mathcal{F}_a(\mathbb{R})$ be a RKHS normed by* (18.7). *Then we have*

$$\int_{\mathbb{R}} |f(x)|^2 \exp(-a^2 x^2)\, dx \leq \frac{\sqrt{2\pi}}{a} \|f\|_{\mathcal{F}_a(\mathbb{R})}{}^2 \tag{18.15}$$

for all $f \in \mathcal{F}_a(\mathbb{R})$. Namely, if we set $W(x) = W_a(x) \equiv \exp(-a^2x^2)$, the restriction operator T from $\mathcal{F}_a(\mathbb{R})$ into $L^2(W_a)$ is bounded with the norm the root square of $\dfrac{\sqrt{2\pi}}{a}$.

Proof. Recall the identity

$$K_a(z,\overline{u}) = \frac{1}{\sqrt{2\pi}a} \exp\left(-\frac{a^2z^2}{2} - \frac{a^2\overline{u}^2}{2}\right) \int_{\mathbb{R}} \exp\left(z\xi + \overline{u}\xi - \frac{\xi^2}{2a^2}\right) d\xi. \tag{18.16}$$

This representation of $K_a(z,\overline{u})$ implies that any $f \in \mathcal{F}_a(\mathbb{R})$ is expressible in the form

$$f(z) = \frac{1}{\sqrt{2\pi}a} \exp\left(-\frac{a^2z^2}{2}\right) \int_{\mathbb{R}} F(\xi) \exp(z\xi) \exp\left(-\frac{\xi^2}{2a^2}\right) d\xi \tag{18.17}$$

for some (of course, uniquely determined) function F satisfying

$$\int_{\mathbb{R}} |F(\xi)|^2 \exp\left(-\frac{\xi^2}{2a^2}\right) d\xi < \infty, \tag{18.18}$$

and we have the isometric identity

$$\|f\|_{\mathcal{F}_a(\mathbb{R})} = \sqrt{\frac{1}{\sqrt{2\pi}a} \int_{\mathbb{R}} |F(\xi)|^2 \exp\left(-\frac{\xi^2}{2a^2}\right) d\xi}. \tag{18.19}$$

Meanwhile, by the Parseval–Plancherel identity, we have, from (18.17)

$$\begin{aligned}
\int_{\mathbb{R}} |f(iy)|^2 \exp(-a^2y^2)dy &= \frac{1}{a^2} \int_{\mathbb{R}} |F(\xi)|^2 \exp\left(-\frac{\xi^2}{a^2}\right) d\xi \\
&\le \frac{1}{a^2} \int_{\mathbb{R}} |F(\xi)|^2 \exp\left(-\frac{\xi^2}{2a^2}\right) d\xi \\
&= \frac{\sqrt{2\pi}}{a} \|f\|_{\mathcal{F}_a(\mathbb{R})}{}^2.
\end{aligned}$$

For $f \in \mathcal{F}_a(\mathbb{R})$, we set $f_1(z) \equiv f(-iz)$. Then, $f_1 \in \mathcal{F}_a(\mathbb{R})$ and $\|f_1\|_{\mathcal{F}_a(\mathbb{R})} = \|f\|_{\mathcal{F}_a(\mathbb{R})}$. Hence, we have the desired result

$$\begin{aligned}
\int_{\mathbb{R}} |f(x)|^2 \exp(-a^2x^2)dx &= \int_{\mathbb{R}} |f_1(ix)|^2 \exp(-a^2x^2)dx \\
&\le \frac{\sqrt{2\pi}}{a} \|f_1\|_{\mathcal{F}_a(\mathbb{R})}{}^2 \\
&= \frac{\sqrt{2\pi}}{a} \|f\|_{\mathcal{F}_a(\mathbb{R})}{}^2.
\end{aligned}$$

Therefore, we have the result. □

We shall determine the condition for the existence of the best approximations $f^* \in \mathcal{F}_a(\mathbb{R})$ of a function $F \in L^2(W_a)$ in the sense

$$\inf_{f \in \mathcal{F}_a(\mathbb{R})} \|Tf - F\|_{L^2(W_a)} = \|Tf^* - F\|_{L^2(W_a)}. \tag{18.20}$$

There exist the best approximations f^* in (18.20) if and only if

$$(T^*F)(z) = \int_{\mathbb{R}} F(\xi) \exp(a^2\xi z) \exp(-a^2\xi^2)\, d\xi \in \mathcal{R}(T^*T) \tag{18.21}$$

and $\mathcal{R}(T^*T)$ is characterized as the RKHS $H_k(E)$ admitting the reproducing kernel

$$\begin{aligned} k(z,\overline{u}) &= (T^*TK_{\overline{u}}, T^*TK_{\overline{z}})_{\mathcal{F}_a(\mathbb{R})} \\ &= (TK_{\overline{u}}, TT^*TK_{\overline{z}})_{L^2(W_a)} \\ &= \frac{\sqrt{\pi}}{a}\int_{\mathbb{R}} \exp\left(a^2\overline{u}\xi\right)\exp\left(\frac{a^2(\xi+z)^2}{4}\right)\exp\left(-a^2\xi^2\right)\,d\xi \\ &= \frac{2\pi}{\sqrt{3}a^2}\exp\left(\frac{1}{3}a^2z^2\right)\exp\left(\frac{1}{3}a^2\overline{u}^2\right)\exp\left(\frac{1}{3}a^2\overline{u}z\right). \end{aligned}$$

Note that the RKHS $H_k(E)$ is composed of all entire functions $f(z)$ with finite norms

$$\|f\|_{H_k(E)} = \frac{a^2}{\sqrt[4]{12\pi^4}}\sqrt{\iint_{\mathbb{C}} |f(z)|^2 \exp\left(-a^2x^2+\frac{a^2y^2}{3}\right)\,dx\,dy}. \tag{18.22}$$

Of course, $\{Tf; f\in\mathcal{F}\}$ and, in particular $\{\exp(a^2\overline{u}\xi); u\in\mathbb{C}\}$ are complete in $L^2(W_a)$ and so $Tf^*=F$ in $L^2(W_a)$ in (18.20). Hence, we have

Proposition 18.3. *For all $F\in L^2(W_a)$, F is realized as an image of $f^*\in\mathcal{F}_a(\mathbb{R})$ by $T:\mathcal{F}_a(\mathbb{R})\to L^2(W)$, if and only if*

$$\iint_{\mathbb{C}}\left|\int_{\mathbb{R}} F(\xi)\exp(a^2\xi z)\exp(-a^2\xi^2)\,d\xi\right|^2 \exp\left(-a^2x^2+\frac{a^2y^2}{3}\right)\,dx\,dy<\infty. \tag{18.23}$$

We can obtain an explicit representation of f^* in terms of F. Of course, f^* is uniquely determined.

Note that

$$\begin{aligned} [T^*T(K_a)_{\overline{u}}](z) &= \langle T(K_a)_{\overline{u}}, T(K_a)_{\overline{z}}\rangle_{L^2(W_a)} \\ &= \frac{\sqrt{\pi}}{a}\exp\left(\frac{a^2z^2}{4}\right)\exp\left(\frac{a^2\overline{u}^2}{4}\right)\exp\left(\frac{a^2\overline{u}z}{2}\right). \end{aligned}$$

Therefore, in particular, we have

Proposition 18.4. *Let $f\in\mathcal{F}_a(\mathbb{R})$ and $z\in\mathbb{C}$. Then we can express $f(z)$ in terms of the trace $f(x), x\in\mathbb{R}$ to the real line in the form*

$$\begin{aligned} f(z) = \frac{a^3}{\sqrt[4]{144\pi^3}}\iint_{\mathbb{C}} &\left(\int_{\mathbb{R}} f(\xi)\exp\left(a^2Z\xi-a^2\xi^2\right)d\xi\right) \\ &\cdot\exp\left(\frac{a^2z^2}{4}+\frac{a^2\overline{Z}}{4}+\frac{a^2z\overline{Z}}{2}-a^2X^2+\frac{a^2}{3}Y^2\right)\,dX\,dY, \end{aligned}$$

where $Z=X+iY$.

For a fixed $F\in L^2(W_a)$, we can construct a sequence $\{f_n\}_{n=0}^{\infty}$ satisfying $f_n\in\mathcal{F}_a(\mathbb{R})$ and

$$\lim_{n\to\infty}\|Tf_n-F\|_{L^2(W_a)}=0. \tag{18.24}$$

See [62], 3.1.4 for the details

19. THE TIKHONOV REGULARIZATION

We shall consider some practical and more concrete representation in the extremal functions involved in the Tikhonov regularization by using the theory of reproducing kernels. Recall that for compact operators the singular values and singular functions representations are popular and in a sense, the representation may be considered complicated.

Furthermore, when $\mathbf{d}$ contains error or noises, error estimates are important. For this fundamental problem, we have the following results:

At first, we need

Proposition 19.1. *Let $L : H_K \to \mathcal{H}$ be a bounded linear operator, and define the inner product*

$$\langle f_1, f_2 \rangle_{H_{K_\alpha}} = \alpha \langle f_1, f_2 \rangle_{H_K} + \langle Lf_1, Lf_2 \rangle_{\mathcal{H}}$$

for $f_1, f_2 \in H_K$. Then $(H_K, \langle \cdot, \cdot \rangle_{H_{K_\alpha}})$ is a reproducing kernel Hilbert space whose reproducing kernel is given by

$$K_\alpha(p,q) = [(\alpha + L^* L)^{-1} K_q](p).$$

Here, $K_\alpha(p,q)$ is the solution $\tilde{K}_\alpha(p,q)$ of the functional equation

$$\tilde{K}_\alpha(p,q) + \frac{1}{\alpha}(L\tilde{K}_q, LK_p)_{\mathcal{H}} = \frac{1}{\alpha} K(p,q), \tag{19.1}$$

that is corresponding to the Fredholm integral equation of the second kind for many concrete cases. Here,

$$\tilde{K}_q = \tilde{K}_\alpha(\cdot, q) \in H_K \quad \text{for} \quad q \in E, \qquad K_p = K(\cdot, p) \quad \text{for} \quad p \in E.$$

Proposition 19.2. *In the Tikhonov functional*

$$f \in H_K \mapsto \left\{ \alpha \|f : H_K\|^2 + \|Lf - \mathbf{d} : \mathcal{H}\|^2 \right\} \in \mathbf{R}$$

attains the minimum and the minimum is attained only at $f_{\mathbf{d},\alpha} \in H_K$ such that

$$(f_{\mathbf{d},\alpha})(p) = \langle \mathbf{d}, LK_\alpha(\cdot, p) \rangle_{\mathcal{H}}.$$

Furthermore, $(f_{\mathbf{d},\alpha})(p)$ satisfies

$$|(f_{\mathbf{d},\alpha})(p)| \leq \sqrt{\frac{K(p,p)}{2\alpha}} \|\mathbf{d}\|_{\mathcal{H}}. \tag{19.2}$$

This proposition means that in order to obtain good approximate solutions, we must take a sufficiently small α, however, here we have restrictions for them, as we see, when $\mathbf{d}$ moves to $\mathbf{d}'$, by considering $f_{\mathbf{d},\alpha}(p) - f_{\mathbf{d}',\alpha}(p)$ in connection with the relation of the difference $\|\mathbf{d} - \mathbf{d}'\|_{\mathcal{H}}$. This fact is a very natural one, because we cannot obtain good solutions from the data containing errors. Here we wish to know how to take a small α a priori and what is the bound for it. These problems are very important practically and delicate ones, and we have many methods.

The basic idea may be given as follows. We examine for various α tending to zero, the corresponding extremal functions. By examining the sequence of the approximate extremal functions, when it converges to some function numerically and after then when the sequence diverges numerically, it will give the bound for α numerically. See [25, 26, 27].

For this important problem and the method of L-curve, see [38, 35], for example.

The Tikhonov regularization is very popular and widely applicable in numerical analysis for its practical power. The application of the theory of reproducing kernels will give more concrete representations of the extremal functions in the Tikhonov regularization.

20. APPROXIMATIONS BY SOBOLEV SPACES BY TIKHONOV REGULARIZATION

We shall give the prototype examples with the first order Sobolev Hilbert space $W^{2,1}(\mathbb{R})$.

For the first order Sobolev Hilbert space $W^{2,1}(\mathbb{R})$ we shall consider the two bounded linear operators $L_1 : W^{2,1}(\mathbb{R}) \mapsto L_1 f \equiv f \in L^2(\mathbb{R})$ and $L_2 : W^{2,1}(\mathbb{R}) \mapsto L_2 f \equiv f' \in L^2(\mathbb{R})$. Then, the associated reproducing kernels $K_{1,1}(x, y; \lambda)$ and $K_{1,2}(x, y; \lambda)$ for the RKHSs with the norms

$$\|f\|_{H_{K_{1,1}(\cdot,\cdot;\lambda)}(\mathbb{R})} = \sqrt{\lambda \|f\|^2_{W^{2,1}(\mathbb{R})} + \|f\|^2_{L^2(\mathbb{R})}} \tag{20.1}$$

and

$$\|f\|_{H_{K_{1,2}(\cdot,\cdot;\lambda)}(\mathbb{R})} = \sqrt{\lambda \|f\|^2_{W^{2,1}(\mathbb{R})} + \|f'\|^2_{L^2(\mathbb{R})}} \tag{20.2}$$

are given by

$$K_{1,1}(x, y; \lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\xi(x-y))}{\lambda\xi^2 + (\lambda+1)} d\xi = \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp\left(-\sqrt{\frac{\lambda+1}{\lambda}}|x-y|\right) \tag{20.3}$$

and

$$K_{1,2}(x, y; \lambda) = \int_{-\infty}^{\infty} \frac{\exp(i\xi(x-y))}{(\lambda+1)\xi^2 + \lambda} \frac{d\xi}{2\pi} = \frac{1}{2\sqrt{\lambda(\lambda+1)}} \exp\left(-\sqrt{\frac{\lambda}{\lambda+1}}|x-y|\right), \tag{20.4}$$

respectively; we can also see directly. Hence, for any $g \in L^2(\mathbb{R})$, the best approximate functions $f^*_{1,1}(x; \lambda, g)$ and $f^*_{1,2}(x; \lambda, g)$ in the senses

$$\begin{aligned} &\inf_{f \in W^{2,1}(\mathbb{R})} \left\{ \lambda \|f\|^2_{W^{2,1}(\mathbb{R})} + \|f - g\|^2_{L^2(\mathbb{R})} \right\} \\ &= \lambda \|f^*_{1,1}(\cdot; \lambda, g)\|^2_{W^{2,1}(\mathbb{R})} + \|f^*_{1,1}(\cdot; \lambda, g) - g\|^2_{L^2(\mathbb{R})} \end{aligned}$$

and

$$\begin{aligned} &\inf_{f \in W^{2,1}(\mathbb{R})} \left\{ \lambda \|f\|^2_{W^{2,1}(\mathbb{R})} + \|f' - g\|^2_{L^2(\mathbb{R})} \right\} \\ &= \lambda \|f^*_{1,2}(\cdot; \lambda, g)\|^2_{W^{2,1}(\mathbb{R})} + \|f^{*\prime}_{1,2}(\cdot; \lambda, g) - g\|^2_{L^2(\mathbb{R})} \end{aligned} \tag{20.5}$$

are given by

$$f^*_{1,1}(x; \lambda, g) = \frac{1}{2\sqrt{\lambda(\lambda+1)}} \int_{\mathbb{R}} g(\xi) \exp\left(-\sqrt{\frac{\lambda+1}{\lambda}}|\xi - x|\right) d\xi \tag{20.6}$$

and

$$f^*_{1,2}(x;\lambda,g) = \frac{1}{2\sqrt{\lambda(\lambda+1)}} \int_{\mathbb{R}} g(\xi) \frac{\partial}{\partial \xi} \exp\left(-\sqrt{\frac{\lambda}{\lambda+1}} |\xi - x| \right) d\xi, \tag{20.7}$$

respectively. Note that $f^*_{1,2}(x;\lambda,g)$ can be considered as an approximate and generalized solution of the differential equation

$$y' = g(x) \quad \text{on} \quad \mathbb{R} \tag{20.8}$$

in the first order Sobolev Hilbert space $W^{2,1}(\mathbb{R})$. We can enjoy many computer graphics with many concrete examples.

Here we consider the typical and elementary differential operator

$$Ly = y'' + \alpha y' + \beta y \tag{20.9}$$

on the whole real line $\mathbb{R}$. For the formal generalizations, results and formulas are similar for higher order ordinary differential equations.

In order to consider a generalized solution of $Ly = g$, for $g \in L^2(\mathbb{R})$, we shall consider naturally the Sobolev space $H^2_S(\mathbb{R})$ on the whole real line $\mathbb{R}$ with finite norms

$$\|f\|_{H^2_S(\mathbb{R})} \equiv \left\{ \int_{-\infty}^{\infty} (f''(x)^2 + 2f'(x)^2 + f(x)^2) dx \right\}^{1/2} < \infty \tag{20.10}$$

admitting the reproducing kernel

$$K(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i\xi(x-y))}{(1+\xi^2)^2} d\xi = \frac{1}{4} e^{-|x-y|}(1+|x-y|). \tag{20.11}$$

Because from (20.9), we can see the existence of the second order derived functions. Now, we consider the best approximation problem; for any given $g \in L^2(\mathbb{R})$ and for any $\lambda > 0$,

$$\inf \left\{ \lambda \|f\|_{H^2_S(\mathbb{R})}{}^2 + \|Lf - g\|_{L^2(\mathbb{R})}{}^2 \, : \, f \in H^2_S(\mathbb{R}) \right\}. \tag{20.12}$$

Then for the RKHS $H_{K_\lambda}(\mathbb{R})$ consisting of all the members of $H^2_S(\mathbb{R})$ with the norm

$$\|f\|_{H_{K_\lambda}(\mathbb{R})} = \sqrt{\lambda \|f\|_{H^2_S(\mathbb{R})}{}^2 + \|Lf\|_{L^2(\mathbb{R})}{}^2}, \tag{20.13}$$

the reproducing kernel $K_\lambda(x,y)$ can be calculated directly by using Fourier's integrals as follows:

$$K_\lambda(x,y) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\exp(i(x-y))\, d\xi}{\lambda(\xi^2+1)^2 + |-\xi^2 + i\alpha\xi + \beta|^2}. \tag{20.14}$$

We thus obtain the member of $H^2_S(\mathbb{R})$ with the minimum norm which attains the infimum (20.12) as follows:

$$f^*_{\lambda,g}(x) = \frac{1}{2\pi} \int_{\mathbb{R}} \left\{ g(\xi) \cdot \int_{\mathbb{R}} \frac{(-\eta^2 - i\alpha\eta + \beta) \exp(-i\eta(\xi - x))}{\lambda(\eta^2+1)^2 + |-\eta^2 + i\alpha\eta + \beta|^2} d\eta \right\} d\xi. \tag{20.15}$$

For $g \in L^2(\mathbb{R})$, if there exists a solution $\hat{f}_g$ of the equation

$$Ly(x) = g(x) \quad \text{on} \quad \mathbb{R}, \tag{20.16}$$

then we have the representation, by using Fourier's integral

$$\hat{f}_g(x) = \frac{1}{2\pi}\int_{\mathbb{R}}\left\{g(\xi)\int_{\mathbb{R}}\frac{\exp(-i\eta(\xi - x))}{-\eta^2 + i\alpha\eta + \beta}\,d\eta\right\}d\xi. \tag{20.17}$$

Then,

$$\lim_{\lambda\downarrow 0} f^*_{\lambda,g}(x) = \hat{f}_g(x), \tag{20.18}$$

uniformly.

We thus obtain a general approximation formula:

Proposition 20.1. *Let $g \in L^2(\mathbb{R})$, $\lambda > 0$ and $s > \frac{1}{2}$. Define*

$$Q_{\lambda,s}(x) \equiv \frac{1}{\pi}\int_0^\infty \frac{(-p^2+\beta)\cos(p\,x) + \alpha\,p\,\sin(p\,x)}{\lambda(p^2+1)^s + p^4 + (\alpha^2 - 2b)p^2 + \beta^2}\,dp \quad (x \in \mathbb{R}) \tag{20.19}$$

and

$$F^*_{\lambda,s,g} \equiv g * Q_{\lambda,s}. \tag{20.20}$$

Then, $F^*_{\lambda,s,g}$ minimizes

$$\min_{F\in H^s(\mathbb{R})}\left(\lambda\,{\|F\|_{H^s(\mathbb{R})}}^2 + {\|g - L(D)F\|_{L^2(\mathbb{R})}}^2\right) \tag{20.21}$$

and the minimizer is unique.

21. GENERAL INHOMOGENEOUS PDEs ON THE WHOLE SPACES

We consider very simple approximate solutions for the general inhomogeneous partial differential equation

$$L(D)u = g \quad \text{on} \quad \mathbb{R}^n\,, \tag{21.1}$$

in the class of the functions of the s-th order Sobolev Hilbert space $H^s(\mathbb{R}^n)$ on the whole real space $\mathbb{R}^n (n \geq 1, s \geq m \geq 1, s > n/2)$, and for any complex-valued $L^2(\mathbb{R}^n)$-function g. Here, $L(D)$ denotes a nontrivial general linear partial differential operator with complex constant coefficients on $\mathbb{R}^n$ of order m. That is, we consider a linear partial differential operator

$$L(D) = \sum_{|\alpha|\leq m} a_\alpha \left(\frac{\partial}{\partial x}\right)^\alpha, \tag{21.2}$$

with the a_α's being complex numbers and there is a multi-index α_0 of length m such that $a_\alpha \neq 0$. Many constant coefficient partial differential equations are under the scope of the main results for inhomogeneous linear partial differential equations with complex constant coefficients of all types on the whole space $\mathbb{R}^n$.

For simplicity, we write

$$\mathbf{L}(\xi) \equiv e^{-ix\cdot\xi}L(D)e^{ix\cdot\xi} \tag{21.3}$$

by using a complex polynomial $\mathbf{L}$. Then we have

Proposition 21.1. *Let $n \geq 1, s \geq m \geq 1$ and $s > n/2$.*

(1) *For any complex-valued function* $g \in L^2$ *and for any* $\lambda > 0$,

$$\inf_{F \in H^s(\mathbb{R}^n)} \left\{ \lambda \|F\|^2_{H^s(\mathbb{R}^n)} + \|g - L(D)F\|^2_{L^2(\mathbb{R}^n)} \right\} \tag{21.4}$$

is attained by a unique function $F^*_{\lambda,s,g}$.

(2) *Let us write*

$$Q_{\lambda,s}(\eta) \equiv \frac{1}{(2\pi)^n} \int_{\mathbb{R}^n} \frac{\overline{\mathbf{L}(p)} \exp(-ip \cdot \eta)}{\lambda(|p|^2+1)^s + |\mathbf{L}(p)|^2} dp \quad (\eta \in \mathbb{R}^n). \tag{21.5}$$

Then, the extremal function $F^*_{\lambda,s,g}$ *is represented by*

$$F^*_{\lambda,s,g} = g * Q_{\lambda,s}. \tag{21.6}$$

(3) *If* g *is expressed as* $g = L(D)F$, *for a function* $F \in H^s(\mathbb{R}^n)$, *then as* $\lambda \downarrow 0$ *we have*

$$F^*_{\lambda,s,g} \to F, \tag{21.7}$$

uniformly.

Let $c > 0$ be a fixed number. The examples are the following:

(1) The $\overline{\partial}$-operator:

$$\frac{\partial}{\partial \overline{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad (x,y) \in \mathbb{R} \times \mathbb{R}. \tag{21.8}$$

(2) The heat operator:

$$\partial_t u - c^2 \Delta_x u \quad (x,t) \in \mathbb{R}^{n-1} \times (0,\infty). \tag{21.9}$$

When we consider the heat operator, we fix $t > 0$.

(3) The wave operator:

$$\partial_t^2 - c^2 \Delta_x \quad (x,t) \in \mathbb{R}^{n-1} \times \mathbb{R}. \tag{21.10}$$

Although there are not global solutions in (21.9), the main result is still applicable.

From concrete examples, we can compute the representations (21.6) and we know the approximate solutions (21.6) converge to the analytical solutions of (21.1) as in (21.7).

22. PDEs AND INVERSE PROBLEMS

We will be able to apply our theory to various inverse problems to look for the whole data from local data of the domain or from some boundary data. Here, we will refer to these problems with a prototype example in order to show this basic idea.

If $F_1 = 0$ and λ is very close to zero then the problem may be transformed into the one that we wish to construct the solution u of the differential equation

$$Lu = 0 \quad \text{on} \quad G \tag{22.1}$$

satisfying

$$Bu = F_2 \quad \text{on} \quad D. \tag{22.2}$$

Our general theory gives a practical construction method for this inverse problem from the observation F_2 on the part D, we construct u on the whole domain G satisfying the equation $Lu = 0$.

We recall a Sobolev embedding theorem [6, pp. 18–19]. In order to use the results in the framework of Hilbert spaces, we assume $p = q = 2$ there.

Let $W_2^\ell(G)$ ($\ell = 0, 1, 2, \ldots$) be the Sobolev-Hilbert space on G, where $G \subset \mathbb{R}^n$ is a bounded domain with a one piecewise-smooth continuously differentiable boundary $\Gamma \equiv \partial G$. We assume that

$$k \geq \ell - \frac{n}{2}. \tag{22.3}$$

Let $m = 0, 1, 2, \ldots$ such that

$$m > n - 2(\ell - k). \tag{22.4}$$

Let $D = D^m \subset G \cup \Gamma$ be any C^ℓ-manifold of dimension m. Then, for any $u \in W_2^\ell(G)$, the derivative $\partial_\alpha u \in L^2(D)(x \in D)$, where $\|\alpha\| \leq k$, and there exists $M > 0$ such that

$$\|\partial_\alpha u\|_{L^2(D)} \leq M\|u\|_{W_2^\ell(G)}, \quad (u \in W_2^\ell(G)). \tag{22.5}$$

Of course

$$\|u\|_{W_2^\ell(G)} \leq \|u\|_{W_2^\ell(\mathbb{R}^n)}, \tag{22.6}$$

and we can construct the reproducing kernel for the space $W_2^\ell(\mathbb{R}^n)$ by using the Fourier integral with $2\ell > n$. Then, for any linear differential operator L with variable coefficients on G satisfying

$$\|Lu\|_{L^2(G)} \lesssim \|u\|_{W_2^\ell(G)} \tag{22.7}$$

and for any linear (boundary) operator B with variable coefficients on D satisfying

$$\|Bu\|_{L^2(D)} \lesssim \|u\|_{W_2^\ell(G)}, \tag{22.8}$$

we can discuss the best approximation: For any $F_1 \in L^2(G)$, for any $F_2 \in L^2(D)$ and for any $\lambda > 0$,

$$\inf_{u \in W_2^\ell(\mathbf{R^n})} \left\{ \lambda\|u\|^2_{W_2^\ell(\mathbb{R}^n)} + \|F_1 - Lu\|_{L^2(G)}{}^2 + \|F_2 - Bu\|_{L^2(D)}{}^2 \right\}. \tag{22.9}$$

For some more recent general discretization principle with many concrete examples, see [15, 16], containing numerical experiments and numerical viewpoints.

For the great work in

Convergence analysis of Tikhonov regularization for non-linear statistical inverse problems,

see [50].

23. PRACTICAL APPLICATIONS TO TYPICAL INVERSE PROBLEMS

At least, until about 20 years ago, we had very difficult inverse problems that are important in some practical problems as follows:

1): Inverse source problem; that is in the Poisson equation

$$\triangle u = -\rho,$$

from the observation of the potential u for the out side of the support ρ, look for the source ρ.

2): The problem in the heat conduction; that is, from some heat $u(x,t)$ observation at a time t, look for the initial heat distribution $u(x,0)$.

3): Real inversion formulas for the Laplace transform.

These problems were indeed difficult in both mathematics and numerical realizations of the solutions and so, they are called ill-posed problems and very difficult problems.

We were able to solve these problems by using the theory of reproducing kernels applying the Tikhonov regularization. However, for the real inversion formula of Laplace transform, we needed the great power of computers by H. Fujiwara. These global theories were published in the book and these are applicable in some general linear problems in the viewpoint of practical problems ([62]). Here, we state their essential parts.

1) Inverse source problem in the Poisson equation ([1])

We obtain very and surprisingly simple approximate solutions for the Poisson equation, for any $L_2(\mathbf{R}^n)$ function g,

$$\Delta u = g \quad \text{on} \quad \mathbf{R}^n \tag{23.1}$$

in the class of the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n (n \geq 1, s \geq 2, s > n/2)$.

We shall use the n order Sobolev Hilbert space H^n comprising functions F on $\mathbf{R}^n$ with the norm (Here, of course, $r_1 + r_2 + \cdots + r_n = \nu$.)

$$\|F\|_{H_S}^2 = \sum_{\nu=0}^{n} {}_nC_\nu \sum_{r_1,r_2,\ldots,r_n \geq 0}^{\nu} \frac{\nu!}{r_1!r_2!\cdots r_n!} \int_{\mathbf{R}^n} \left(\frac{\partial^\nu F(x)}{\partial x_1^{r_1} \partial x_2^{r_2} \cdots \partial x_n^{r_n}} \right)^2 dx. \tag{23.2}$$

This Hilbert space admits the reproducing kernel

$$K(x,y) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{1}{(1+|\xi|^2)^n} e^{i(x-y)\cdot\xi} d\xi \tag{23.3}$$

as we see easily by using Fourier's transform. Note that the Sobolev Hilbert space H^s admitting the reproducing kernel (22.3) for $n = s$ can be defined for any positive number s in term of Fourier integrals $\hat{F}$ of F

$$\hat{F}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbf{R}^n} e^{-i\xi\cdot x} F(x) dx$$

as follows:

$$\|F\|_{H^s}^2 = \int_{\mathbf{R}^n} |\hat{F}(\xi)|^2 (1+|\xi|^2)^s d\xi$$

for any $s > n/2$.

Under these conditions our formulations and results are stated as follows:

Proposition 23.1. *Let $n \geq 1, s \geq 2$ and $s > n/2$. For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F^*_{\lambda,s,g}$ in the sense*

$$\inf_{F \in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|g - \Delta F\|_{L_2(\mathbf{R}^n)}^2 \right\} = \lambda \|F^*_{\lambda,s,g}\|_{H^s}^2 + \|g - \Delta F^*_{\lambda,s,g}\|_{L_2(\mathbf{R}^n)}^2 \tag{23.4}$$

exists uniquely and $F^*_{\lambda,s,g}$ is represented by

$$F^*_{\lambda,s,g}(x) = \int_{\mathbf{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi \tag{23.5}$$

for

$$Q_{\lambda,s}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{-|p|^2 e^{-ip\cdot(\xi - x)} dp}{\lambda(|p|^2+1)^s + |p|^4}. \tag{23.6}$$

If, for $F \in H^s$ we consider the solution $u_F(x)$: $\Delta u_F(x) = F(x)$ and we take $u_F(\xi)$ as g, then we have the favourable result: as $\lambda \to 0$

$$F^*_{\lambda,s,g} \to F, \tag{23.7}$$

uniformly.

2) The problem in the heat conduction ([2]).

From some heat $u(x,t)$ observation at a time t, we shall look for the initial heat $u(x,0)$.

We gave simple approximate real inversion formulas for the Gaussian convolution (the Weierstrass transform)

$$u_F(x,t) = (L_t F)(x) = \frac{1}{(4\pi t)^{n/2}} \int_{\mathbf{R}^n} F(\xi) \exp\left\{ -\frac{|\xi - x|^2}{4t} \right\} d\xi \tag{23.8}$$

for the functions of the s order Sobolev Hilbert space H^s on the whole real space $\mathbf{R}^n (n \geq 1, s > n/2)$. This integral transform which represents the solution $u(x,t)$ of the heat equation

$$u_t(x,t) = u_{xx}(x,t) \quad \text{on} \quad \mathbf{R}^n \times \{t > 0\} \quad (u(x,0) = F(x) \quad \text{on} \quad \mathbf{R}^n). \tag{23.9}$$

In this problem we will use the same norm and reproducing kernel as (23.2) and (23.3).

Under those situations our formulations and results are stated as follows:

Proposition 23.2. *For any function $g \in L_2(\mathbf{R}^n)$ and for any $\lambda > 0$, the best approximate function $F^*_{\lambda,s,g}$ in the sense*

$$\inf_{F \in H^s} \left\{ \lambda \|F\|_{H^s}^2 + \|g - u_F(\cdot,t)\|_{L_2(\mathbf{R}^n)}^2 \right\}$$
$$= \lambda \|F^*_{\lambda,s,g}\|_{H^s}^2 + \|g - u_{F^*_{\lambda,s,g}}(\cdot,t)\|_{L_2(\mathbf{R}^n)}^2 \tag{23.10}$$

exists uniquely and $F^*_{\lambda,s,g}$ *is represented by*

$$F^*_{\lambda,s,g}(x) = \int_{\mathbf{R}^n} g(\xi) Q_{\lambda,s}(\xi - x) d\xi \tag{23.11}$$

for

$$Q_{\lambda,s}(\xi - x) = \frac{1}{(2\pi)^n} \int_{\mathbf{R}^n} \frac{e^{-ip\cdot(\xi-x)}dp}{\lambda(|p|^2+1)^s e^{|p|^2 t} + e^{-|p|^2 t}}. \tag{23.12}$$

If, for $F \in H^s$ *we consider the output* $u_F(x,t)$ *and we take* $u_F(\xi,t)$ *as* g*, then we have the favourable result: as* $\lambda \to 0$

$$F^*_{\lambda,s,g} \to F, \tag{23.13}$$

uniformly.

3) Real inversion formulas for the Laplace transform ([3])

We obtained a very natural and numerical real inversion formula of the Laplace transform

$$(\mathcal{L}F)(p) = f(p) = \int_0^\infty e^{-pt} F(t) dt, \quad p > 0 \tag{23.14}$$

for functions F of some natural function space. The inversion of the Laplace transform is, in general, given by a complex form, however, we are interested in and are requested to obtain its real inversion in many practical problems. However, the real inversion will be very involved and one might think that its real inversion will be essentially involved, because we must catch "analyticity" from the real or discrete data.

We shall introduce the simple reproducing kernel Hilbert space (RKHS) H_K comprised of absolutely continuous functions F on the positive real line $\mathbf{R}^+$ with finite norms

$$\left\{ \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt \right\}^{1/2} \quad (\ F(0) = 0 \ ). \tag{23.15}$$

This Hilbert space admits the reproducing kernel

$$K(t,t') = \int_0^{\min(t,t')} \xi e^{-\xi} d\xi. \tag{23.16}$$

Then we see that

$$\int_0^\infty |(\mathcal{L}F)(p)p|^2 dp \le \frac{1}{2} \|F\|^2_{H_K}; \tag{23.17}$$

that is, the linear operator on H_K,$(\mathcal{L}F)(p)p$ into $L_2(\mathbf{R}^+, dp) = L_2(\mathbf{R}^+)$ is bounded ([4]). For the reproducing kernel Hilbert spaces H_K satisfying (23.15), we can find some general spaces. Therefore, from the general theory, we obtain

Proposition 23.3. *For any* $g \in L_2(\mathbf{R}^+)$ *and for any* $\alpha > 0$*, the best approximation* $F^*_{\alpha,g}$ *in the sense*

$$\inf_{F \in H_K} \left\{ \alpha \int_0^\infty |F'(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F)(p)p - g\|^2_{L_2(\mathbf{R}^+)} \right\}$$

$$= \alpha \int_0^\infty |F^{*\prime}_{\alpha,g}(t)|^2 \frac{1}{t} e^t dt + \|(\mathcal{L}F^*_{\alpha,g})(p)p - g\|^2_{L_2(\mathbf{R}^+)} \tag{23.18}$$

exists uniquely and we obtain the representation

$$F^*_{\alpha,g}(t) = \int_0^\infty g(\xi) \left(\mathcal{L}K_\alpha(\cdot,t)\right)(\xi)\xi d\xi. \tag{23.19}$$

Here, $K_\alpha(\cdot,t)$ is determined by the functional equation

$$K_\alpha(t,t') = \frac{1}{\alpha}K(t,t') - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}K_t)(p)p)_{L_2(\mathbf{R}^+)} \tag{23.20}$$

for $K_{\alpha,t'} = K_\alpha(\cdot,t')$ and $K_t = K(\cdot,t)$.

We shall look for the approximate inversion $F^*_{\alpha,g}(t)$ by using (23.19). For this purpose, we take the Laplace transfrom of (23.20) in t and change the variables t and t' as in

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi) = \frac{1}{\alpha}(\mathcal{L}K(\cdot,t'))(\xi) - \frac{1}{\alpha}((\mathcal{L}K_{\alpha,t'})(p)p, (\mathcal{L}(\mathcal{L}K_t)(p)p))(\xi))_{L_2(\mathbf{R}^+)}. \tag{23.21}$$

Note that

$$K(t,t') = \begin{cases} -te^{-t} - e^{-t} + 1 & \text{for} \quad t \le t' \\ -t'e^{-t'} - e^{-t'} + 1 & \text{for} \quad t \ge t'. \end{cases}$$

$$(\mathcal{L}K_\alpha(\cdot,t))(\xi) = e^{-t'p}e^{-t'}\left[\frac{-t'}{p(p+1)} + \frac{-1}{p(p+1)^2}\right] + \frac{1}{p(p+1)^2}. \tag{23.22}$$

$$\int_0^\infty e^{-qt'}(\mathcal{L}K(\cdot,t'))(p)dt' = \frac{1}{pq(p+q+1)^2}. \tag{23.23}$$

Therefore, by setting $(\mathcal{L}K_\alpha(\cdot,t))(\xi)\xi = H_\alpha(\xi,t)$, which is needed in (22.19), we obtain the Fredholm integral equation of the second type

$$\alpha H_\alpha(\xi,t) + \int_0^\infty H_\alpha(p,t)\frac{1}{(p+\xi+1)^2}dp = -\frac{e^{-t\xi}e^{-t}}{\xi+1}\left(t + \frac{1}{\xi+1}\right) + \frac{1}{(\xi+1)^2}. \tag{23.24}$$

By solving this integral equation, H. Fujiwara derived a very reasonable numerical inversion formula for the integral transform and he expanded very good algorithms for numerical and real inversion formulas of the Laplace transform. For more detailed references and comments for this equation, see [25, 26, 27].

In particular, H. Fujiwara solved the integral equation (23.24) with 6000 points discretization with *600 digits precision* based on the concept of *infinite precision* which is in turn based on *multiple-precision arithmetic*. Then, the regularization parameters were $\alpha = 10^{-100}, 10^{-400}$ surprisingly. For the integral equation, he used the *DE formula* by H. Takahashi and M. Mori, using double exponential transforms. H. Fujiwara was successful in deriving numerically the inversion for the Laplace transform of the distribution delta which was proposed by V. V. Kryzhniy as a difficult case. This fact will mean that the above results valid for very general functions approximated by the functions of the reproducing kernel Hilbert space $H_K(\mathbf{R}^+)$.

Fujiwara made the software foe the real inversion of the Laplace tansform and we can use it with his helpful guide. It was our purpose of our numerical challenges.

We showed many Figures for the numerical experiments in the complete version [28] by Professor H. Fujiwara. For the heat conduction problem, we gave the results in [45].

24. NUMERICAL EXPERIMENTS

1) Inverse source problem in the Poisson equation ([1])

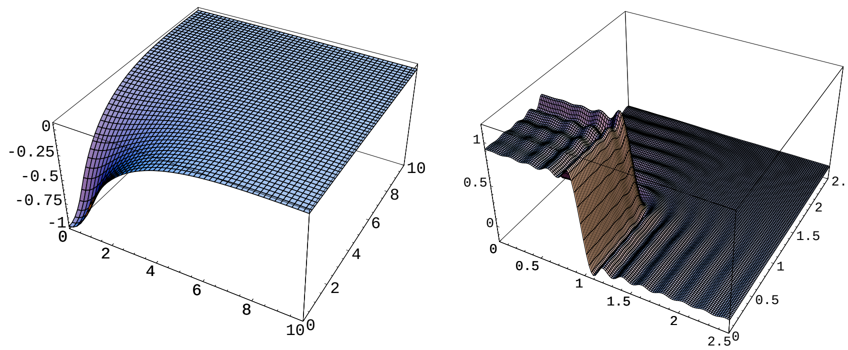

FIGURE 1. For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,2,g}(x_1, x_2)$ and $\Delta F^*_{\lambda,2,g}(x_1, x_2)$ for $\lambda = 10^{-2}$.

This numerical result shows that the new method is working effectively and is useful.

2) The problem in the heat conduction ([2])

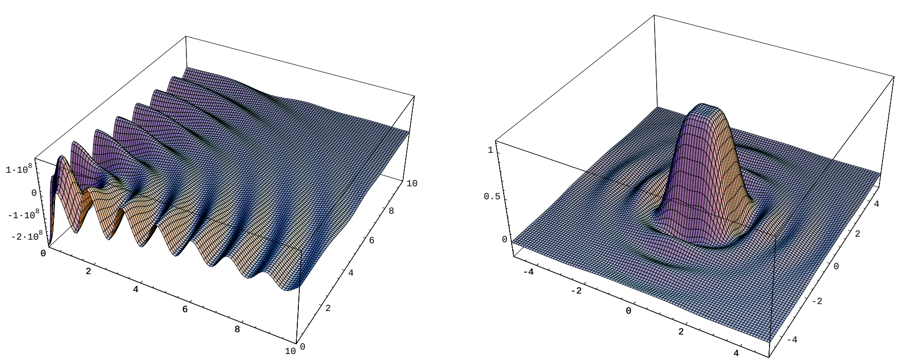

FIGURE 2. For $g(x_1, x_2) = \chi_{[-1,1]}(x_1) \times \chi_{[-1,1]}(x_2)$ on $\mathbf{R}^2$, the figures of $F^*_{\lambda,s,g}(x_1, x_2)$ and $u_{F^*_{\lambda,s,g}(x_1,x_2;t)}$ for $t = 1, s = 2, \lambda = 10^{-22}$.

The results of this numerical experiment prove the usefulness and correctness of our method.

3) Real inversion formulas for the Laplace transform ([3])

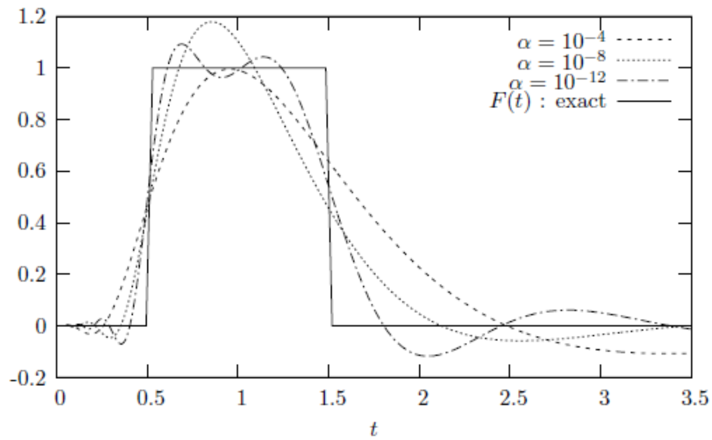

FIGURE 3. For $F(t) = \chi(t, [1/2, 3/2])$, the characteristic function and for $\alpha = 10^{-4}, 10^{-8}, 10^{-12}$.

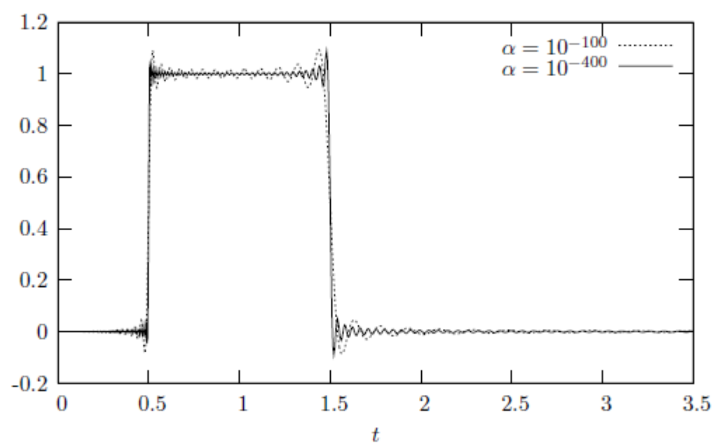

FIGURE 4. For $F(t) = \chi(t, [1/2, 3/2])$, the characteristic function and for $\alpha = 10^{-100}, 10^{-400}$.

The results of these numerical experiments show that our method is effective even when there are jumps in the target function, and in Figure 4 we use a high-precision numerical algorithm developed by our collaborator Professor Fujiwara.

With the great impact of Fujiwara, we considered and gave many concrete results constructing the results by using finite number of data, directly with a unified method using the theory of reproducing kernels in the book [62]. Now we think for the regularized solutions, we can obtain good realizations by computers.

REMARKS

As basic general concepts of the applications of reproducing kernels, we consider more as in

General Fractional Functions - Division by Zero,
Convolutions, Integral Transforms and Integral Equations,
Operator Equations With a Parameter,
Sampling Theory, Kramer -Type Lemma and Loss Error,
Membership Problems for RKHSs,
Graphs and Reproducing Kernels,
Natural Outputs and Global Inputs of Linear Systems,
Identifications of Nonlinear Systems,
Band Preserving and Phase Retrieval,
Singular Integral Equations and Reproducing Kernels,

and others except analytic function theory. For some global theory by the second author in complex analysis, see [54, 55]. For a global theory, see the book [62]. Any essence of the theory of reproducing kernels by the authors is a definite one and so the contents of this paper is, in part, overlapping the contents of the second author's plenary lecture at the ISAAC 2015, Macau Congress [61]. The materials are contained in the definite book [62].

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TSUTOMU MATSUURA: GUNMA UNIVERSITY CENTER FOR MATHEMATICS AND DATA SCIENCE, SHOWA-MACHI 3-39-22, MAEBASHI 376-8515, JAPAN
Email address: matsuura@gunma-u.ac.jp

SABUROU SAITOH: INSTITUTE OF REPRODUCING KERNELS, KIRYU 376-0041, JAPAN
Email address: saburou.saitoh@gmail.com