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ON THE SOLUTION OF THE ILL-POSED CAUCHY PROBLEM FOR ELLIPTIC SYSTEMS OF THE FIRST ORDER

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Abstract. In this paper, we consider the problem of recovering solutions of matrix factorizations of the Helmholtz equation in a four-dimensional bounded domain from their values on a part of the boundary of this domain, i.e., the Cauchy problem. Based on the Carleman function, an explicit solution of the Cauchy problem for matrix factorizations of the Helmholtz equation is constructed.

1. Introduction

The theory of ill-posed problems originated in an unusual way. As a rule, a new concept is a subject in which its creator takes a keen interest. The concept of ill-posed problems was introduced by Hadamard with the comment that these problems are physically meaningless and not worthy of the attention of serious researchers. Despite Hadamard's pessimistic forecasts, however, his unloved "child" has turned into a powerful theory whose results are used in many fields of pure and applied mathematics. What is the secret of its success? The answer is clear. Ill-posed problems occur everywhere and it is unreasonable to ignore them.

Many problems of an applied nature, such as geo- and biophysical, electrodynamics, gas-, hydro- and aerodynamic, problems of plasma physics, etc., are reduced to the equations of mathematical physics. In fact, the very construction of an equation of mathematical physics, which adequately describes certain physical laws of the world around us, is a solution to a certain problem, which it is natural to call "inverse" The researcher observes a phenomenon and tries to construct an equation whose solution has the observed properties. Usually, the resulting equations are based on physical laws that allow us to formulate the general form of differential relations. As a rule, they contain a certain number of arbitrary functions that determine the properties of the physical medium. If the properties of the medium are known, then the equation of mathematical physics, combined with the boundary and initial conditions, makes it possible to predict the development of a physical phenomenon in the space-time region. This is a

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classic problem for the equations of mathematical physics. In the theory of inverse problems such problems are called "direct". In modern natural science, the following inverse problems very often arise: the general form of the equation of mathematical physics is known, but the characteristic properties of the medium are not known, they must be determined from the observed solutions of the equation. A typical situation is when direct measurements inside a certain area are impossible for one reason or another, however, indirect observation and qualitative and quantitative measurements of physical fields at the boundary or outside this area are possible. In mathematical terms, such problems should ensure the correctness of the problem statement. Boundary and initial conditions are formulated in order to select the desired one from the set of possible solutions of a differential equation with partial derivatives. These additional conditions should not be very many (solutions must exist) and not very few (there should not be many solutions). This is related to the concept of a well-posed problem statement. The concept of the correctness of a problem statement in mathematical physics was formulated at the beginning of the 20th century by the famous French mathematician J. Hadamard [11]. The necessity to solve nonstationary problems similar to that presented above requires more exact determination of problem solution. In problems conditionally well-posed according to Tikhonov, we have to do not just with a solution, but with a solution that belongs to some class of solutions. Making the class of admissible solutions narrower allows one in some cases to pass to a well-posed problem.

We say that a problem is Tikhonov well-posed if:

1) the solution of the problem is a priori known to exist in some class;

2) in this class, the solution is unique;

3) the solution continuously depends on input data.

The fundamental difference here consists in separating out the class of admissible solutions. The classes of a priori restrictions differ widely. In consideration of ill-posed problems, the problem statement itself undergoes substantial changes: the condition that the solution belongs to a certain set is to be included into the problem statement. J. Hadamard showed this on the example of the Cauchy problem for the Laplace equation, which has become a classic example of an ill-posed problem. The need to consider problems of mathematical physics that are incorrect in the classical sense (according to Hadamard) in connection with the problems of interpreting geophysical observational data was first indicated in 1943 by the twice Hero of Socialist Labor, Academician of the USSR Academy of Sciences A.N. Tikhonov [50]. He showed that if the class of possible solutions is reduced to a compact set, then the existence and uniqueness of the solution implies its stability. Ways of development of the theory and methods for solving ill-posed problems are associated with the names of prominent mathematicians A.N. Tikhonov, M.M. Lavrentiev, V.K. Ivanov, as well as with the scientific mathematical schools they created, which largely determined the development of theories and applications of ill-posed problems. A large number of problems in mathematical physics that do not satisfy the Hadamard correctness conditions are reduced to an operator equation of the first kind. Since the problems of mathematical physics describe real processes in nature, they must satisfy certain requirements. The stability requirement means that any physically defined process must continuously depend on the initial and boundary conditions and on the inhomogeneous term in the equation, i.e. should be characterized by functions that change little with small changes in the initial data. Such processes are not physically defined. Stability is also important for the approximate solution of problems. Among mathematical problems, a class of problems stands out, the solutions of which are unstable to small changes in the initial data. They are characterized by the fact that arbitrarily small changes in the initial data can lead to arbitrarily large changes in the solutions. Problems of this type are, in essence, ill-posed. They belong to the class of ill-posed problems.

In the last three decades, regularization methods have been proposed for solving ill-posed problems. However, in a significant part of the work, deterministic methods are used to introduce a priori information both about the solution itself and about the errors in the initial data of the problem. Unreasonably little attention is paid to the choice of optimal values for the parameters of algorithms, which would allow obtaining solutions with the smallest error, as well as the construction of algorithms with given accuracy characteristics. There are no effective algorithms that allow taking into account the available a priori information about the desired solution (for example, about the range of possible values of the coefficient of the identified model). The lack of software developed in the environment of a universal mathematical package (for example, Mathcad) creates significant difficulties for engineers and experimenters (who are not programmers) in using regularizing algorithms in practice. Tasks that do not satisfy all of the above requirements 1) - 3) are, according to Hadamard, incorrectly delivered. In 1926, T. Carleman (see, for instance [7], p. 41) constructed a formula that connects the values of the analytic function of a complex variable at the points of the region with its values on a piece of the boundary of this region. The construction of the Carleman function makes it possible in these problems to construct a regularization and obtain an estimate of the conditional stability. It is known that the Helmholtz equation in different spaces has a fundamentally different solution. In the future, using the construction of constructing a fundamental solution, we will construct an approximate solution for the Helmholtz equation. M.M. Lavrent'ev, in his works on the Cauchy problem for the Laplace equation and for some other ill-posed problems of mathematical physics, indicated a method for distinguishing the correctness class and developed stable methods for solving them (see. for instance [45]-[46]). M.M. Lavrent'ev proposed the construction of a regularized solution of the Cauchy problem for the Laplace equation using the Carleman function. Moreover, in the 1977s, Sh. Yarmukhamedov pointed out the construction of a family of fundamental solutions parametrized by an entire function with certain properties [7]. This construction is used to construct explicit formulas that restore solutions of elliptic equations in a domain from their Cauchy data on a piece of the domain boundary. Such formulas are also called Carleman formulas. The multidimensional Carleman formula was constructed by L.A. Aizenberg [11]. In unstable problems, the image of the operator is not closed, therefore, the solvability condition cannot be written in terms of continuous linear functionals. So, in the Cauchy problem for elliptic equations with data on a part of the boundary of a domain, the solution is usually unique, the problem is solvable for an everywhere dense data set, but this set is not closed. Consequently, the theory of solvability of such problems is much more difficult and deeper than the theory of solvability of the Fredholm equations. The first results in this direction appeared only in the mid-1980s in the works of L.A. Aizenberg, A.M. Kytmanov and N.N. Tarkhanov [10]. An analogue of the Carleman formula for one class of elliptic systems with constant coefficients on the plane is considered in the work of E.V. Arbuzov and A.L. Bukhgeim [12]. The construction of the Carleman matrix for elliptic systems was carried out by Sh. Yarmukhamedov, N.N. Tarkhanov, I.E. Niyozov and others. In papers [17]-[33] and [55] The questions of exact and approximate solutions of the ill-posed Cauchy problem for various factorizations of the Helmholtz equations are studied. Such problems arise in mathematical physics and in various fields of natural science (for example, in electro-geological exploration, in cardiology, in electrodynamics, etc.). Using the construction of previous works, the validity of the fundamental solution for the matrix factorization of the Helmholtz equation in various spaces was proved in the works [17]-[33] and [55]. Currently, the theory of ill-posed problems is one of the topical problems of partial differential equations.

Many scientific and applied problems, studied at the world level, in many cases are reduced to the study of ill-posed boundary value problems for partial differential equations. Applied research on conditional correctness and construction of an approximate solution for given values on a part of the boundary of the region, for equations of elliptical type, are especially important in hydrodynamics, geophysics and electrodynamics. The study of a family of regularizing solutions to ill-posed problems served as an impetus for the beginning of studies of the well-posedness class when narrowed to a compact set. Therefore, the study of ill-posed problems for linear elliptic systems of the first order is one of the topical problems in the theory of partial differential equations. At present, in the world, in the study of ill-posed boundary value problems for linear elliptic systems of the first order, the construction of a regularized solution plays a special role. The Cauchy problem for elliptic equations is ill-posed (example Hadamard, see for instance [11], p. 39). Boundary problems, as well as numerical solutions of some problems, are considered in works [2], [6], [10], [15]-[16], [43], and [47].

At present, special attention is paid to topical aspects of differential equations and mathematical physics, which have scientific and practical applications in the fundamental sciences. In particular, special attention is paid to the study of various ill-posed boundary value problems for partial differential equations of elliptic type, which have practical application in applied sciences. As a result, significant results were obtained in studies of ill-posed boundary value problems for partial differential equations, that is, approximate solutions were constructed using Carleman matrices in explicit form from approximate data in special domains, estimates of conditional stability and solvability criteria were established. The first results, from the point of view of practical importance, for ill-posed problems and for reducing the class of possible solutions to a compact set and reducing problems to stable ones were obtained in the works of A.N. Tikhonov (see [50]). In the works of M.M. Lavrent'ev, estimates were obtained that characterize the stability of the spatial problem in the class of bounded solutions of the Cauchy problem for the

Laplace equation and some other ill-posed problems of mathematical physics in a straight cylinder, as well as for an arbitrary spatial domain with a sufficiently smooth boundary (see, for instance [45]-[46]).

In this work, based on the results of works [45]-[46], [51]-[54], based on the Cauchy problem for the Laplace and Helmholtz equations, an explicit Carleman matrix was constructed and, on its basis, a regularized solution of the Cauchy problem for the matrix factorization of the Helmholtz equation. In work [12], the calculation of double integrals with the help of some connection between wave equation and ODE system was considered. The problem of one nonlocal boundary value problem for a loaded parabolic-hyperbolic equation with three lines of type change was considered in work [14].

The problem of reconstructing the solution for matrix factorization of the Helmholtz equation (see, for instance [17], [18], [19], [20], [21], [22], [23], [24], [25], [26], [27], [28], [29], [30], [31], [32] and [33]), is one of the topical problems in the theory of differential equations.

At present, there is still interest in classical ill-posed problems of mathematical physics. This direction in the study of the properties of solutions of Cauchy problem for Laplace equation was started in [7], [45]-[46], [5], [51]-[54] and subsequently developed in [3]-[9], [48]-[49], [17]-[42], [55].

2. Basic information and statement of the Cauchy problem

Let $\mathfrak{R}^4$ be a four-dimensional real Euclidean space,

$$\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \mathfrak{R}^4, \quad \eta = (\eta_1, \eta_2, \eta_3, \eta_4) \in \mathfrak{R}^4,$$

$$\xi' = (x_1, \xi_2, x_3) \in \mathfrak{R}^3, \quad \eta' = (\eta_1, \eta_2, \eta_3) \in \mathfrak{R}^3.$$

We introduce the following notation:

$$\rho = |\eta - \xi|, \quad \alpha = |\eta' - \xi'|, \quad z = i\tau\sqrt{\vartheta^2 + \alpha^2} + \beta, \quad w_0 = i\tau\alpha + \beta,$$

$$\beta = \tau\eta_4, \quad \tau = tg\frac{\pi}{2r}, \quad r > 1, \quad \vartheta \geq 0, \quad s = \alpha^2,$$

$$\Xi_r = \{\eta: \quad |\eta'| < \tau\eta_4, \quad \eta_4 > 0\}, \quad \partial\Xi_r = \{\eta: \quad |\eta'| = \tau\eta_4, \quad \eta_4 > 0\},$$

$$\frac{\partial}{\partial\xi} = \left(\frac{\partial}{\partial\xi_1}, \frac{\partial}{\partial\xi_2}, \frac{\partial}{\partial\xi_3}, \frac{\partial}{\partial\xi_4}\right)^T, \; \frac{\partial}{\partial\xi} = \chi^T, \; \chi^T = \begin{pmatrix} \chi_1 \\ \chi_2 \\ \chi_3 \\ \chi_4 \end{pmatrix}\text{-transposed vector } \chi,$$

$$\mathfrak{U}(\xi) = (\mathfrak{U}_1(\xi), \ldots, \mathfrak{U}_n(\xi))^T, \quad \vartheta^0 = (1, \ldots, 1) \in \mathfrak{R}^n, \quad n = 2^4,$$

$$\mathcal{E}(w) = \begin{Vmatrix} w_1 \ldots 0 \\ \ldots\ldots. \\ 0 \ldots w_n \end{Vmatrix}\text{- diagonal matrix, } w = (w_1, ..., w_n) \in \mathfrak{R}^n.$$

$\Xi_r \subset \mathfrak{R}^4$ be a bounded simply-connected domain, the boundary of which consists of the surface of the cone $\partial\Xi_r$, and a smooth piece of the surface Σ, lying in the cone Ξ_r, i.e., $\partial\Xi_r = \Sigma \bigcup \Upsilon$, $\quad \Upsilon = \partial\Xi_r \backslash \Sigma$. Let $(0, 0, ..., \xi_4) \in \Xi_r$, $\xi_4 > 0$.

Let $\mathfrak{D}(\chi^T)$ be a $(n \times n)-$ dimensional matrix with elements consisting of a set of linear functions with constant coefficients of the complex plane for which the following condition is satisfied:

$$\mathfrak{D}^*(\chi^T)\mathfrak{D}(\chi^T) = \mathcal{E}((|\chi|^2 + \Lambda^2)\vartheta^0),$$

where $\mathfrak{D}^*(\chi^T)$ is the Hermitian conjugate matrix $\mathfrak{D}(\chi^T)$, $\Lambda-$ is a real number.

We consider a system of differential equations in the region Ξ

$$\mathfrak{D}\left(\frac{\partial}{\partial \xi}\right)\mathfrak{U}(\xi) = 0, \tag{2.1}$$

where $\mathfrak{D}\left(\frac{\partial}{\partial \xi}\right)$ is the matrix of first-order differential operators.

We denote by $\mathcal{A}(\Xi_r)$ the class of vector functions in the domain Ξ_r continuous on $\overline{\Xi}_r = \Xi_r \bigcup \partial\Xi_r$ and satisfying system (2.1).

3. Construction of the Carleman matrix and the Cauchy problem

Formulation of the problem. Suppose $\mathfrak{U}(\eta) \in \mathcal{A}(\Xi_r)$ and

$$\mathfrak{U}(\eta)|_{\Sigma} = \mathfrak{f}(\eta), \quad \eta \in \Sigma. \tag{3.1}$$

Here, $\mathfrak{f}(\eta)$ a given continuous vector-function on Σ. It is required to restore the vector function $\mathfrak{U}(\eta)$ in the domain Ξ_r, based on it's values $\mathfrak{f}(\eta)$ on Σ.

If $\mathfrak{U}(\eta) \in \mathcal{A}(\Xi_r)$, then the following integral formula of Cauchy type is valid

$$\mathfrak{U}(\xi) = \int\limits_{\partial\Xi_r} \mathfrak{N}(\eta, \xi; \Lambda)\mathfrak{U}(\eta)ds_\eta, \quad \xi \in \Xi, \tag{3.2}$$

where

$$\mathfrak{N}(\eta, \xi; \Lambda) = \left(\mathcal{E}\left(\varphi_4(\Lambda\rho)\vartheta^0\right)\mathfrak{D}^*\left(\frac{\partial}{\partial \xi}\right)\right)\mathfrak{D}(\nu^T).$$

Here $\nu = (\nu_1, \nu_2, \nu_3, \nu_4)-$is the unit exterior normal, drawn at a point η, the surface $\partial\Xi_r$, $\varphi_4(\Lambda\rho)-$ is the fundamental solution of the Helmholtz equation in $\mathfrak{R}^4$ where $\varphi_4(\Lambda\rho)$ defined by the following formula:

$$\begin{gathered}\varphi_4(\Lambda\rho) = \mathcal{P}_4\Lambda\frac{\mathcal{H}_1^{(1)}(\Lambda\rho)}{\rho}, \\ \mathcal{P}_4 = \frac{1}{4i\pi}.\end{gathered} \tag{3.3}$$

Here $\mathcal{H}_1^{(1)}(\Lambda\rho)-$ is the Hankel function of the first kind of $1-$ th order (see, for instance [44]).

We denote by $\mathfrak{K}(z)$ is an entire function taking real values for real z, ($z = u + iv, \quad u, v-$real numbers) and satisfying the following conditions:

$$\begin{gathered}\mathfrak{K}(\vartheta) \neq 0, \quad \sup_{v \geq 1}\left|v^p\mathfrak{K}^{(p)}(z)\right| = B(\vartheta, p) < \infty, \\ -\infty < \vartheta < \infty, \quad p = 0, 1, 2, 3, 4.\end{gathered} \tag{3.4}$$

We define the function $\Psi(\eta, \xi; \Lambda)$ at $\eta \neq \xi$ by the following equality

$$\Psi(\eta, \xi; \Lambda) = \frac{1}{c_4 \mathfrak{K}(\xi_4)} \frac{\partial}{\partial s} \int_0^\infty \mathfrak{Im} \left[\frac{\mathfrak{K}(z)}{z - \xi_4} \right] \frac{\vartheta \mathfrak{I}_0(\Lambda\vartheta)}{\sqrt{\vartheta^2 + \alpha^2}} d\vartheta, \tag{3.5}$$

where $c_4 = -2\omega_4$; $\mathfrak{I}_0(\Lambda\xi) = \mathfrak{J}_0(i\Lambda\vartheta)-$is the Bessel function of the first kind of zero order (see, [5]), ω_4- area of a unit sphere in space $\mathfrak{R}^4$.

In the formula (3.5), choosing

$$\mathfrak{K}(z) = E_r(\mu^{1/r} z), \quad \mathfrak{K}(\xi_4) = E_r(\mu^{1/r} \gamma), \quad \gamma = \tau\xi_4, \quad \mu > 0, \tag{3.6}$$

we get

$$\Psi_\mu(\eta, \xi; \Lambda) = \frac{E_r(\mu^{1/r}\gamma)}{c_4} \frac{\partial}{\partial s} \int_0^\infty \mathfrak{Im} \left[\frac{E_r(\mu^{1/r} z)}{z - \xi_4} \right] \frac{\vartheta \mathfrak{I}_0(\Lambda\vartheta)}{\sqrt{\vartheta^2 + \alpha^2}} d\vartheta. \tag{3.7}$$

Here $E_r(\mu^{1/r} z)-$ is the entire Mittag-Leffler function (see, [8]). In [1], using the S-generalized beta function, a new generalization of the Mittag-Leffler function and its properties is presented.

The formula (3.2) is true if instead $\varphi_4(\Lambda\rho)$ of substituting the function

$$\Psi_\mu(\eta, \xi; \Lambda) = \varphi_4(\Lambda\rho) + \Pi_\mu(\eta, \xi; \Lambda), \tag{3.8}$$

where $\Pi_\mu(\eta, \xi; \Lambda)-$ is the regular solution of the Helmholtz equation with respect to the variable η, including the point $\eta = \xi$.

Then the integral formula has the form:

$$\mathfrak{U}(\xi) = \int_{\partial\Xi_r} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta, \quad \xi \in \Xi, \tag{3.9}$$

where

$$\mathfrak{N}(\eta, \xi; \Lambda) = \left(\mathcal{E} \left(\Psi_\mu(\eta, \xi; \Lambda)\vartheta^0 \right) \mathfrak{D}^* \left(\frac{\partial}{\partial \xi} \right) \right) \mathfrak{D}(\nu^T).$$

Recall the basic properties of the Mittag-Leffler function. The entire function of Mittag-Leffler is defined by a series.

$$\sum_{n=1}^\infty \frac{z^n}{\Gamma(1 + r^{-1}n)} = E_r(z), \quad z = u + iv,$$

where $\Gamma(s)-$ is the Euler gamma function.

We denote by $\gamma_\varepsilon(\beta_0)(\varepsilon > 0, \quad 0 < \beta_0 < \pi)$ the contour in the complex plane ζ, run in the direction of non-decreasing $\arg \zeta$ and consisting of the following parts:
1. The beam $\arg \zeta = -\beta_0, \quad |\zeta| \geq \varepsilon$;
2. The arc $-\beta_0 < \arg \zeta < \beta_0$ of circle $|\zeta| = \varepsilon$;
3. The beam $\arg \zeta = \beta_0, \quad |\zeta| \geq \varepsilon$.

The contour $\gamma_\varepsilon(\beta_0)$ divides the plane ζ into two unbounded simply connected domains Ξ_r^- and Ξ_r^+ lying to the left and to the right of $\gamma_\varepsilon(\beta_0)$, respectively.

Let $r > 1, \quad \dfrac{\pi}{2r} < \beta_0 < \dfrac{\pi}{r}$.

Denote
$$\psi_r(z) = \frac{1}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^r)}{\zeta - z} d\zeta, \tag{3.10}$$
Then the following integral representations are valid:
$$E_r(z) = \psi_r(z), \quad \xi \in \Xi_r^-, \tag{3.11}$$
$$E_r(z) = r\exp(z^r) + \psi_r(z), \quad \xi \in \Xi_r^+, \tag{3.12}$$
From these formulas we find
$$\left.\begin{array}{l} |E_r(z)| \leq r\exp(\operatorname{Re} z^r) + |\psi_r(z)|, \quad |\arg z| \leq \frac{\pi}{2r} + \ell_0, \\ |E_r(z)| \leq |\psi_r(z)|, \quad \frac{\pi}{2r} + \ell_0 \leq |\arg z| \leq \pi, \quad \ell_0 > 0 \end{array}\right\} \tag{3.13}$$
$$|\psi_r(z)| \leq \frac{M}{1 + |z|}, \quad M = const \tag{3.14}$$
$$E_r(z) \approx r\exp(z^r), \quad z > 0, \quad z \to \infty, \tag{3.15}$$
Further, since $E_r(z)$ is real with real z, then
$$\operatorname{Re}\psi_r(z) = \frac{r}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{2\zeta - \operatorname{Re} z}{(\zeta - z)\zeta - \overline{z})} \exp(\zeta^r) d\zeta,$$
$$\operatorname{Im}\psi_r(z) = \frac{r\operatorname{Im}(z)}{2\pi i} \int\limits_{\gamma_\varepsilon(\beta_0)} \frac{\exp(\zeta^r)}{(\zeta - z)\zeta - \overline{z})} d\zeta,$$
The information given here concerning the function $E_r(z)$ is taken from (see, for instance [19] and [23]).

In what follows, to prove the main theorems, we need the following estimates for the function $\Psi_\mu(\eta, \xi; \Lambda)$.

Lemma 3.1. *Let* $\xi = (\xi_1, \xi_2, \xi_3, \xi_4) \in \Xi_r, \quad \eta \neq \xi, \quad \mu \geq \Lambda + \mu_0, \quad \mu_0 > 0$, *then*

1) at $\beta \leq \alpha$ *inequalities are satisfied*
$$|\Psi_\mu(\eta, \xi; \Lambda)| \leq \mathcal{K}(r, \Lambda) \frac{\mu}{\rho^2} \exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r, \tag{3.16}$$
$$\left| \frac{\partial \Psi_\mu(\eta, \xi; \Lambda)}{\partial \eta_j} \right| \leq \mathcal{K}(r, \Lambda) \frac{\mu^4}{\rho^3} \exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,4}. \tag{3.17}$$
$$\left| \frac{\partial \Psi_\mu(\eta, \xi; \Lambda)}{\partial \xi_j} \right| \leq \mathcal{K}(r, \Lambda) \frac{\mu^4}{\rho^3} \exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,4}. \tag{3.18}$$
2) at $\beta > \alpha$ *inequalities are satisfied*
$$|\Psi_\mu(\eta, \xi; \Lambda)| \leq \mathcal{K}(r, \Lambda) \frac{\mu}{\rho^2} \exp(-\mu\gamma^r + \mu\operatorname{Re} z_0^r), \quad \mu > 1, \quad \xi \in \Xi_r, \tag{3.19}$$
$$\left| \frac{\partial \Psi_\mu(\eta, \xi; \Lambda)}{\partial \eta_j} \right| \leq \mathcal{K}(r, \Lambda) \frac{\mu}{\rho^2} \exp(-\mu\gamma^r + \mu\operatorname{Re} z_0^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,4}. \tag{3.20}$$
$$\left| \frac{\partial \Psi_\mu(\eta, \xi; \Lambda)}{\partial \xi_j} \right| \leq \mathcal{K}(r, \Lambda) \frac{\mu}{\rho^2} \exp(-\mu\gamma^r + \mu\operatorname{Re} z_0^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,4}. \tag{3.21}$$

Here $\mathcal{K}(r,\Lambda)$ is the function depending on r and Λ.

For a fixed $\xi \in \Xi_r$ we denote by Σ^* the part of Σ on which $\beta \geq \alpha$. If $\xi \in \Xi_r$, then $\Sigma = \Sigma^*$ (in this case, $\beta = \tau\eta_4$ and the inequality $\beta \geq \alpha$ means that η lies inside or on the surface cone).

4. The continuation formula and regularization according to M.M. Lavrent'ev's

Theorem 4.1. *Let* $\mathfrak{U}(\eta) \in \mathcal{A}(\Xi_r)$ *it satisfy the inequality*

$$|\mathfrak{U}(\eta)| \leq M, \quad \eta \in \Upsilon = \partial\Xi_r \backslash \Sigma^*. \tag{4.1}$$

If

$$\mathfrak{U}_\mu(\xi) = \int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta, \quad \xi \in \Xi_r, \tag{4.2}$$

then the following estimates are true

$$|\mathfrak{U}(\xi) - \mathfrak{U}_\mu(\xi)| \leq M\mathcal{K}_r(\Lambda,\xi)\mu^2\exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{4.3}$$

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j} - \frac{\partial\mathfrak{U}_\mu(\xi)}{\partial\xi_j}\right| \leq M\mathcal{K}_r(\Lambda,\xi)\mu^2\exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,4}. \tag{4.4}$$

Here and below functions bounded on compact subsets of the domain Ξ_r, we denote by $\mathcal{K}_r(\Lambda,\xi)$.

Proof. Let us first estimate inequality (4.3). Using the integral formula (3.9) and the equality (4.2), we obtain

$$\mathfrak{U}(\xi) = \int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta + \int\limits_{\partial\Xi_r\backslash\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta =$$

$$= \mathfrak{U}_\mu(\xi) + \int\limits_{\partial\Xi_r\backslash\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta, \quad \xi \in \Xi_r.$$

Taking into account the inequality (4.1), we estimate the following

$$\begin{aligned} |\mathfrak{U}(\xi) - \mathfrak{U}_\mu(\xi)| &\leq \left|\int\limits_{\partial\Xi_r\backslash\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta\right| \leq \\ &\leq \int\limits_{\partial\Xi_r\backslash\Sigma^*} |\mathfrak{N}_\mu(\eta,\xi;\Lambda)|\,|\mathfrak{U}(\eta)|\,ds_\eta \leq M\int\limits_{\partial\Xi_r\backslash\Sigma^*} |\mathfrak{N}_\mu(\eta,\xi;\Lambda)|\,ds_\eta, \quad \xi \in \Xi_r. \end{aligned} \tag{4.5}$$

We estimate the integrals $\int\limits_{\partial\Xi_r\backslash\Sigma^*} |\Psi_\mu(\eta,\xi;\Lambda)|\,ds_\eta$, $\int\limits_{\partial\Xi_r\backslash\Sigma^*} \left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_j}\right| ds_\eta$ and $\int\limits_{\partial\Xi_r\backslash\Sigma^*} \left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta$ on the part $\partial\Xi_r\backslash\Sigma^*$ of the plane $\eta_4 = 0$ ($j = \overline{1,3}$).

Separating the imaginary part of (3.7), we obtain

$$\Psi_\mu(\eta,\xi;\Lambda) = \frac{E_r(\mu^{1/r}\gamma)}{c_4}\left[\frac{\partial}{\partial s}\int\limits_0^\infty \frac{(\eta_4-\xi_4)\mathfrak{Im}E_r(\mu^{1/r}z)}{\vartheta^2+\rho^2}\,\frac{\vartheta\mathfrak{I}_0(\Lambda\vartheta)}{\sqrt{\vartheta^2+\alpha^2}}d\vartheta - \right.$$
$$\left. -\frac{\partial}{\partial s}\int\limits_0^\infty \frac{\vartheta\mathfrak{Re}\,E_r(\mu^{1/r}z)}{\vartheta^2+\rho^2}\mathfrak{I}_0(\Lambda\vartheta)\,d\vartheta\right], \ \eta\neq\xi, \quad \xi_4>0. \tag{4.6}$$

Given (4.6) and the inequality

$$\mathfrak{I}_0(\Lambda\vartheta) \le \sqrt{\frac{2}{\Lambda\pi\vartheta}}, \tag{4.7}$$

we have

$$\int\limits_{\partial\Xi_r\setminus\Sigma^*} |\Psi_\mu(\eta,\xi;\Lambda)|\,ds_y \le \mathcal{K}_r(\Lambda,\xi)\mu^2\exp(-\mu\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r, \tag{4.8}$$

To estimate the second integral, we use the equality

$$\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_j} = \frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial s}\frac{\partial s}{\partial\eta_j} = 2(\eta_j-\xi_j)\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial s},$$
$$s=\alpha^2, \quad j=\overline{1,3} \tag{4.9}$$

Given equality (4.6), inequality (4.7) and equality (4.9), we obtain

$$\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_j}\right| ds_\eta \le \mathcal{K}_r(\Lambda,\xi)\mu^2\exp(-\mu\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r,$$
$$j=\overline{1,3}. \tag{4.10}$$

Now, we estimate the integral $\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta.$

Taking into account equality (4.6) and inequality (4.7), we obtain

$$\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta \le \mathcal{K}_r(\Lambda,\xi)\mu^2\exp(-\mu\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r. \tag{4.11}$$

From inequalities (4.8), (4.10), (4.11) and (4.5), we obtain an estimate (4.3).

Now let us prove inequality (4.4). To do this, we take the derivatives from equalities (3.9) and (4.2) with respect to ξ_j, $j=\overline{1,4}$, then we obtain the following:

$$\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j} = \int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta + \int\limits_{\partial\Xi_r\setminus\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta,$$
$$\frac{\partial\mathfrak{U}_\mu(\xi)}{\partial\xi_j} = \int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta, \quad \xi\in\Xi_r, \quad j=\overline{1,4}. \tag{4.12}$$

Taking into account the (4.12) and inequality (4.1), we estimate the following

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}-\frac{\partial_\mu\mathfrak{U}(\xi)}{\partial\xi_j}\right|\le\left|\int\limits_{\partial\Xi_r\setminus\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right|\le$$

$$\le\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right||\mathfrak{U}(\eta)|\,ds_\eta\le M\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta,\tag{4.13}$$

$$\xi\in\Xi_r,\quad j=\overline{1,4}.$$

To do this, we estimate the integrals $\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta,\quad(j=\overline{1,3})$ and $\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta$ on the part $\partial\Xi_r\setminus\Sigma^*$ of the plane $\eta_4=0$.

To estimate the first integrals, we use the equality

$$\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}=\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial s}\frac{\partial s}{\partial\xi_j}=-2(\eta_j-\xi_j)\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial s},$$
$$s=\alpha^2,\quad j=1,3\tag{4.14}$$

Given equality (4.6), inequality (4.7) and equality (4.14), we obtain

$$\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta\le\mathcal{K}_r(\mu,\xi)\mu^2\exp(-\mu\gamma^r),\quad\mu>1,\quad\xi\in\Xi_r,$$
$$j=\overline{1,3}.\tag{4.15}$$

Now, we estimate the integral $\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta.$

Taking into account equality (4.6) and inequality (4.7), we obtain

$$\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta\le\mathcal{K}_r(\mu,\xi)\mu^2\exp(-\mu\gamma^r),\quad\mu>1,\quad\xi\in\Xi_r.\tag{4.16}$$

From inequalities (4.13), (4.15) and (4.16), we obtain an estimate (4.4).
Theorem 4.1 is proved. □

Corollary 4.2. *For each $x\in\Xi_r$, the equalities are true*

$$\lim_{\mu\to\infty}\mathfrak{U}_\mu(\xi)=\mathfrak{U}(\xi),\quad\lim_{\mu\to\infty}\frac{\partial\mathfrak{U}_\mu(\xi)}{\partial\xi_j}=\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j},\quad j=\overline{1,4}.$$

We denote by $\overline{\Xi}_\varepsilon$ the set

$$\overline{\Xi}_\varepsilon=\left\{(\xi_1,\xi_2,\xi_3,\xi_4)\in\Xi_r,\quad a>\xi_4\ge\varepsilon,\quad a=\max_{\Upsilon}\psi(\xi'),\quad 0<\varepsilon<a\right\}.$$

Here, $\psi(\xi')$ - is a surface. It is easy to see that the set $\overline{\Xi}_\varepsilon\subset\Xi_r$ is compact.

Corollary 4.3. *If $x \in \overline{\Xi}_\varepsilon$, then the families of functions $\{\mathfrak{U}_\mu(\xi)\}$ and $\left\{\dfrac{\partial \mathfrak{U}_\mu(\xi)}{\partial \xi_j}\right\}$ converge uniformly for $\mu \to \infty$, i.e.:*

$$\mathfrak{U}_\mu(\xi) \rightrightarrows \mathfrak{U}(\xi), \quad \frac{\partial \mathfrak{U}_\mu(\xi)}{\partial \xi_j} \rightrightarrows \frac{\partial \mathfrak{U}(\xi)}{\partial \xi_j}, \quad j = \overline{1, 4}.$$

It should be noted that the set $E_\varepsilon = \Xi_\rho \backslash \overline{\Xi}_\varepsilon$ serves as a boundary layer for this problem, as in the theory of singular perturbations, where there is no uniform convergence.

5. Estimation of the stability of the solution to the Cauchy problem

Suppose that the surface Σ is given by the equation

$$\eta_4 = \psi(\eta'), \quad \eta' \in \mathfrak{R}^3,$$

where $\psi(\eta')$ is a single-valued function satisfying the Lyapunov conditions.

We put

$$a = \max_{\Upsilon} \psi(\eta'), \quad b = \max_{\Upsilon} \sqrt{1 + \psi'^2(\eta')}.$$

Theorem 5.1. *Let $\mathfrak{U}(\eta) \in \mathcal{A}(\Xi_r)$ satisfy condition (4.1), and on a smooth surface Σ the inequality*

$$|\mathfrak{U}(\eta)| \leq \varsigma, \quad 0 < \varsigma < 1. \tag{5.1}$$

Then the following estimates are true

$$|\mathfrak{U}(\xi)| \leq \mathcal{K}_r(\Lambda, \xi) \mu^2 M^{1-\left(\frac{\gamma}{a}\right)^r} \varsigma^{\left(\frac{\gamma}{a}\right)^r}, \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.2}$$

$$\left|\frac{\partial \mathfrak{U}(\xi)}{\partial \xi_j}\right| \leq \mathcal{K}_r(\Lambda, \xi) \mu^2 M^{1-\left(\frac{\gamma}{a}\right)^r} \varsigma^{\left(\frac{\gamma}{a}\right)^r}, \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1, 4}. \tag{5.3}$$

Here is $a^r = \max\limits_{\eta \in \Sigma} \mathfrak{Re}\, z_0^r$.

Proof. Let us first estimate inequality (5.2). Using the integral formula (3.9), we have

$$\mathfrak{U}(\xi) = \int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta + \int\limits_{\partial \Xi_r \backslash \Sigma^*} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta, \quad \xi \in \Xi_r. \tag{5.4}$$

We estimate the following

$$|\mathfrak{U}(x)| \leq \left|\int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta\right| + \left|\int\limits_{\partial \Xi_r \backslash \Sigma^*} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta\right|, \quad \xi \in \Xi_r. \tag{5.5}$$

Given inequality (5.1), we estimate the first integral of inequality (5.5).

$$\begin{aligned} \left|\int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta, \xi; \Lambda) \mathfrak{U}(\eta) ds_\eta\right| &\leq \int\limits_{\Sigma^*} |\mathfrak{N}_\mu(\eta, \xi; \Lambda)| \, |\mathfrak{U}(\eta)| \, ds_\eta \leq \\ &\leq \varsigma \int\limits_{\Sigma^*} |\mathfrak{N}_\mu(\eta, \xi; \Lambda)| \, ds_\eta, \quad \xi \in \Xi_r. \end{aligned} \tag{5.6}$$

We estimate the integrals $\int\limits_{\Sigma^*} |\Psi_\mu(\eta,\xi;\Lambda)|\, ds_\eta$, $\int\limits_{\Sigma^*} \left|\dfrac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_j}\right| ds_\eta$, $(j = \overline{1,3})$ and $\int\limits_{\Sigma^*} \left|\dfrac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta$ on a smooth surface Σ.

Given equality (4.6) and the inequality (4.7), we have

$$\int\limits_{\Sigma^*} |\Psi_\mu(\eta,\xi;\Lambda)|\, ds_\eta \leq \mathcal{K}_r(\lambda,\xi)\mu^2 \exp\mu(\tau^r a^r - \gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.7}$$

To estimate the second integral, using equalities (4.6) and (4.9) as well as inequality (4.7), we obtain

$$\int\limits_{\Sigma^*} \left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_j}\right| ds_\eta \leq \mathcal{K}_r(\lambda,\xi)\mu^2 \exp\mu(\tau^r a^r - \gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r, \quad j = \overline{1,3}. \tag{5.8}$$

To estimate the integral $\int\limits_{\Sigma^*} \left|\dfrac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta$, using equality (4.6) and inequality (4.7), we obtain

$$\int\limits_{\Sigma^*} \left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\eta_4}\right| ds_\eta \leq \mathcal{K}_r(\lambda,\xi)\mu^2 \exp\mu(\tau^r a^r - \gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.9}$$

From (5.7) - (5.9), we obtain

$$\left|\int\limits_{\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta\right| \leq \mathcal{K}_r(\lambda,\xi)\mu^2\varsigma \exp\mu(\tau^r a^r - \gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.10}$$

The following is known

$$\left|\int\limits_{\partial\Xi_r\setminus\Sigma^*} \mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta\right| \leq \mathcal{K}_r(\Lambda,\xi)\mu^2 M \exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.11}$$

Now taking into account (5.10) - (5.11), we have

$$|\mathfrak{U}(\xi)| \leq \frac{\mathcal{K}_r(\Lambda,\xi)\mu^2}{2}(\varsigma \exp(\mu\tau^r a^r) + M) \exp(-\mu\gamma^r), \quad \mu > 1, \quad \xi \in \Xi_r. \tag{5.12}$$

Choosing ς from the equality

$$\mu = \frac{1}{a^r} \ln\frac{M}{\varsigma}, \tag{5.13}$$

we obtain an estimate (5.2).

Now let us prove inequality (5.3). To do this, we find the partial derivative from the integral formula (3.9) with respect to the variable ξ_j, $\quad j=\overline{1,3}$:

$$\begin{gathered}\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}=\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta+\\ +\frac{\partial\mathfrak{U}_\mu(\xi)}{\partial\xi_j}+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta,\quad \xi\in\Xi_r,\quad j=\overline{1,4}.\end{gathered}\tag{5.14}$$

Here

$$\frac{\partial\mathfrak{U}_r(\xi)}{\partial\xi_j}=\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta.\tag{5.15}$$

We estimate the following

$$\begin{gathered}\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}\right|\le\left|\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right|+\left|\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial x_j}\mathfrak{U}(\eta)ds_\eta\right|\le\\ \le\left|\frac{\partial\mathfrak{U}_\mu(x)}{\partial\xi_j}\right|+\left|\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right|,\quad \xi\in\Xi_r,\quad j=\overline{1,4}.\end{gathered}\tag{5.16}$$

Given inequality (5.1), we estimate the first integral of inequality (5.16).

$$\begin{gathered}\left|\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right|\le\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right||\mathfrak{U}(\eta)|\,ds_\eta\le\\ \le\varsigma\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta,\quad \xi\in\Xi_r,\quad j=\overline{1,4}.\end{gathered}\tag{5.17}$$

To do this, we estimate the integrals $\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta$, $\quad (j=\overline{1,3})$ and $\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta$ on a smooth surface Σ.

Given equality (4.6), inequality (4.7) and equality (4.14), we obtain

$$\begin{gathered}\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta\le\mathcal{K}_r(\Lambda,\xi)\mu^2\exp\mu(\tau^r a^r-\gamma^r),\quad \mu>1,\quad \xi\in\Xi_r,\\ j=\overline{1,3}.\end{gathered}\tag{5.18}$$

Now, we estimate the integral $\int\limits_{\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta$.

Taking into account equality (4.6) and inequality (4.7), we obtain

$$\int\limits_{\Sigma^*}\left|\frac{\partial\Psi_\mu(\eta,\xi;\Lambda)}{\partial\xi_4}\right|ds_\eta \le \mathcal{K}_r(\Lambda,\xi)\mu^2\exp\mu(\tau^r a^r-\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r. \tag{5.19}$$

From (5.18) - (5.19), we obtain

$$\left|\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)\right| \le \mathcal{K}_r(\Lambda,\xi)\mu^2\varsigma\exp\mu(\tau^r a^r-\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r, \quad j=\overline{1,4}. \tag{5.20}$$

The following is known

$$\left|\int\limits_{\partial\Xi_\rho\setminus\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right| \le \mathcal{K}_r(\Lambda,\xi)\mu^2 M\exp(-\mu\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r, \quad j=\overline{1,4}. \tag{5.21}$$

Now taking into account (5.20) - (5.21), we have

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}\right| \le \frac{\mathcal{K}_r(\Lambda,\xi)\mu^2}{2}(\varsigma\exp(\mu\tau^r a^r)+M)\exp(-\mu\gamma^r), \quad \mu>1, \quad \xi\in\Xi_r, \quad j=\overline{1,4}. \tag{5.22}$$

Choosing μ from the equality (5.13) we obtain an estimate (5.3).
Theorem 5.1 is proved. □

Let $\mathfrak{U}(\eta)\in\mathcal{A}(\Xi_r)$ and instead $\mathfrak{U}(\eta)$ on Σ with its approximation $\mathfrak{f}_\varsigma(\eta)$, respectively, with an error $0<\varsigma<1$,

$$\max_{\Sigma}|\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)|\le\varsigma. \tag{5.23}$$

We put

$$\mathfrak{U}_{\mu(\varsigma)}(\xi)=\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{f}_\varsigma(\eta)ds_\eta, \quad \xi\in\Xi_r. \tag{5.24}$$

Theorem 5.2. *Let $\mathfrak{U}(\eta)\in\mathcal{A}(\Xi_r)$ on the part of the plane $\eta_4=0$ satisfy condition* (4.1).

Then the following estimates is true

$$\left|\mathfrak{U}(\xi)-\mathfrak{U}_{\mu(\varsigma)}(\xi)\right|\le\mathcal{K}_r(\Lambda,\xi)\mu^2 M^{1-\left(\frac{\gamma}{a}\right)^r}\varsigma^{\left(\frac{\gamma}{a}\right)^r}, \quad \mu>1, \quad \xi\in\Xi_r. \tag{5.25}$$

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}-\frac{\partial\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}\right|\le\mathcal{K}_r(\Lambda,\xi)\mu^2 M^{1-\left(\frac{\gamma}{a}\right)^r}\varsigma^{\left(\frac{\gamma}{a}\right)^r}, \quad \mu>1, \quad \xi\in\Xi_r, \quad j=\overline{1,4}. \tag{5.26}$$

Proof. From the integral formulas (3.9) and (5.24), we have

$$\mathfrak{U}(\xi)-\mathfrak{U}_{\mu(\mu)}(\xi)=\int\limits_{\partial\Xi_r}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta-\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta=$$

$$=\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta-\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta=$$

$$=\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\left\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\right\}ds_\eta+\int\limits_{\partial\Xi_\rho\backslash\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta.$$

and

$$\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}-\frac{\partial\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}=\int\limits_{\partial\Xi_r}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta-\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial xi_j}\mathfrak{f}_\varsigma(\eta)ds_\eta=$$

$$=\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta-$$

$$-\int\limits_{S^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{f}_\varsigma(\eta)ds_\eta=\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\left\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\right\}ds_\eta+$$

$$+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta,\quad j=\overline{1,4}.$$

Using conditions (4.1) and (5.23), we estimate the following:

$$\left|\mathfrak{U}(\xi)-\mathfrak{U}_{\mu(\varsigma)}(\xi)\right|=\left|\int\limits_{\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\left\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\right\}ds_\eta\right|+$$

$$+\left|\int\limits_{\partial\Xi_r\backslash\Sigma^*}\mathfrak{N}_\mu(\eta,\xi;\Lambda)\mathfrak{U}(\eta)ds_\eta\right|\le\int\limits_{\Sigma^*}\left|\mathfrak{N}_\mu(\eta,\xi;\Lambda)\right|\left|\left\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\right\}\right|ds_\eta+$$

$$+\int\limits_{\partial\Xi_r\backslash\Sigma^*}\left|\mathfrak{N}_\mu(\eta,\xi;\Lambda)\right|\left|\mathfrak{U}(\eta)\right|ds_\eta\le\varsigma\int\limits_{\Sigma^*}\left|\mathfrak{N}_\mu(\eta,\xi;\Lambda)\right|ds_\eta+$$

$$+M\int\limits_{\partial\Xi_r\backslash\Sigma^*}\left|\mathfrak{N}_\mu(\eta,\xi;\Lambda)\right|ds_\eta.$$

and

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}-\frac{\partial\mathfrak{U}_{\mu(\delta)}(x)}{\partial\xi_j}\right|=\left|\int\limits_{\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\left\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\right\}ds_\eta\right|+$$

$$+\left|\int\limits_{\partial\Xi_r\setminus\Sigma^*}\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\mathfrak{U}(\eta)ds_\eta\right|\le\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|\left|\{\mathfrak{U}(\eta)-\mathfrak{f}_\varsigma(\eta)\}\right|ds_\eta+$$

$$+\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|\left|\mathfrak{U}(\eta)\right|ds_\eta\le\varsigma\int\limits_{\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta+$$

$$+M\int\limits_{\partial\Xi_r\setminus\Sigma^*}\left|\frac{\partial\mathfrak{N}_\mu(\eta,\xi;\Lambda)}{\partial\xi_j}\right|ds_\eta,\quad j=\overline{1,4}.$$

Now, repeating the proof of Theorems 4.1 and 5.1, we obtain

$$\left|\mathfrak{U}(\xi)-\mathfrak{U}_{\mu(\varsigma)}(\xi)\right|\le\frac{\mathcal{K}_r(\Lambda,\xi)\mu^2}{2}(\varsigma\exp(\mu\tau^r a^r)+M)\exp(-\mu\gamma^r),$$

$$\left|\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j}-\frac{\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}\right|\le\frac{\mathcal{K}_r(\Lambda,\xi)\mu^2}{2}(\varsigma\exp(\mu\tau^r a^r)+M)\exp(-\mu\gamma^r),\quad j=\overline{1,4}.$$

From here, choosing μ from equality (5.13), we obtain an estimates (5.25) and (5.26).

Theorem 5.2 is proved. □

Corollary 5.3. *For each $\xi\in\Xi_r$, the equalities are true*

$$\lim_{\varsigma\to 0}\mathfrak{U}_{\mu(\varsigma)}(\eta)=\mathfrak{U}(\xi),\quad \lim_{\varsigma\to 0}\frac{\partial\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}=\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j},\quad j=\overline{1,4}.$$

Corollary 5.4. *If $\xi\in\overline{\Xi}_\varepsilon$, then the families of functions $\left\{\mathfrak{U}_{\mu(\varsigma)}(\xi)\right\}$ and $\left\{\frac{\partial\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}\right\}$ converge uniformly for $\varsigma\to 0$, i.e.:*

$$\mathfrak{U}_{\mu(\varsigma)}(\xi)\rightrightarrows\mathfrak{U}(\xi),\quad\frac{\partial\mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial\xi_j}\rightrightarrows\frac{\partial\mathfrak{U}(\xi)}{\partial\xi_j},\quad j=\overline{1,4}.$$

6. Conclusion

Hadamard believed that any mathematical problem corresponding to any physical or technical problem should be correct, since it is difficult to imagine what physical interpretation the solution can have if arbitrarily small changes in the initial data can correspond to large changes in the solution. This called into question the expediency of studying ill-posed problems (examples are given by Hadamard himself). Later it was established that widespread mathematical problems are unstable in certain metrics: the solution of integral equations of the first kind; differentiation of functions known approximately; numerical summation of Fourier series when their coefficient is known approximately; solving systems of linear algebraic equations under conditions of a system determinant close to zero; the Cauchy problem for the Laplace equation; analytic continuation of functions; inverse problems of gravimetry; minimization of functionals; some problems of linear programming and optimal control, as well as optimal design (synthesis of antennas and other physical systems); object control described by differential equations.

This article obtained the following results:

Using the Carleman function, a formula is obtained for the continuation of the solution of linear elliptic systems of the first order with constant coefficients in a spatial bounded domain $\mathfrak{R}^4$. The resulting formula is an analogue of the classical formula of B. Riemann, W. Voltaire and J. Hadamard, which they constructed to solve the Cauchy problem in the theory of hyperbolic equations. An estimate of the stability of the solution of the Cauchy problem in the classical sense for matrix factorizations of the Helmholtz equation is given. The problem is considered in which instead of the exact data of the Cauchy problem; their approximations with a given deviation in the uniform metric are given and under the assumption that the solution of the Cauchy problem is bounded on part Υ of the boundary of the domain Ξ_r; an explicit regularization formula is obtained.

We note that when solving applied problems, one should find the approximate values of $U(\xi)$ and $\dfrac{\partial \mathfrak{U}(\xi)}{\partial \xi_j}$, $\xi \in \Xi_r$, $j = \overline{1,4}$. In this paper, we construct a family of vector-functions $\mathfrak{U}(\xi, \mathfrak{f}_\varsigma) = \mathfrak{U}_{\mu(\varsigma)}(\xi)$ and $\dfrac{\partial \mathfrak{U}(\xi, \mathfrak{f}_\varsigma)}{\partial \xi_j} = \dfrac{\partial \mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial \xi_j}$, $j = \overline{1,4}$ depending on a parameter μ, and prove that under certain conditions and a special choice of the parameter $\mu = \mu(\varsigma)$, at $\varsigma \to 0$, the family $\mathfrak{U}_{\mu(\varsigma)}(\xi)$ and $\dfrac{\partial \mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial \xi_j}$ converges in the usual sense to a solution $\mathfrak{U}(\xi)$ and its derivative $\dfrac{\partial \mathfrak{U}(\xi)}{\partial \xi_j}$ at a point $\xi \in \Xi_r$.

Following A.N. Tikhonov (see [50]), a family of vector-valued functions $\mathfrak{U}_{\mu(\varsigma)}(\xi)$ and $\dfrac{\partial \mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial \xi_j}$ is called a regularized solution of the problem. A regularized solution determines a stable method of approximate solution of the problem.

Thus, functionals $\mathfrak{U}_{\mu(\varsigma)}(\xi)$ and $\dfrac{\partial \mathfrak{U}_{\mu(\varsigma)}(\xi)}{\partial \xi_j}$ determines the regularization of the solution of problem (2.1)–(3.1).

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