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HISTORY OF THE DIVISION BY ZERO AND DIVISION BY ZERO CALCULUS

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Abstract. Division by zero has a long and mysterious history since the origins of mathematics by Euclid and Brahmagupta. We will see that they are very important in mathematics, however they had the serious problems; that is, on the point at infinity and the division by zero, respectively. Indeed, in Euclidean geometry, the point at infinity was vague and meanwhile, in the arithmetic laws of Brahmagupta, the division by zero $1/0$ was impossible. By some new definition of the division by zero $1/0$, we can obtain the new concept of some completion of Euclidean geometry and we can consider some natural division by zero. The concept of division by zero will create the concept of division by zero calculus and this concept will give great impacts to elementary mathematics. In this paper, we will present some essential history of the division by zero with some up-to-date situation. In order to see simply the new results of the division by zero, we will show the typical results in the fundamental objects. We give the fundamental properties of the division by zero calculus.

1. Global history on division by zero

The global history of the division by zero is given by H. G. Romig ([52]) in details.

In short,

A. D. Brahmagupta (628): in general, no quotient, however, $0/0 = 0$.
Bhaskara (1152): $1/0 = \infty$.
John Wallis (1657) said that zero is no number and but$1/0 = \infty$, and he is the first to use the symbol ∞ for infinity.
John Craig (1716): impossible.
Isaac Newton (1744): the integral of dx/x is infinity.
Wolgang Boyai (1831): a/b has no meaning.
Martin Ohm (1832): should not be considered.
De. Morgan (1831): $1/0 = \infty$.
Rudolf Lipschtz (1877): not permissble.
Axel Harnack (1881): impossible.

Meanwhile, note that Euler stated that $1/0 = \infty$ ([19]). See the details:

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Dividing by Nothing by Alberto Martinez:

> Title page of Leonhard Euler, Vollständige Anleitung zur Algebra, Vol. 1 (edition of 1771, first published in 1770), and p. 34 from Article 83, where Euler explains why a number divide by zero gives infinity. https://notevenpast.org/dividing-nothing/

N. Abel used 1/0 as a notation of INFINITY: https://ja.wikipedia.org/wiki/

For the paper [52], C. B. Boyer ([9]) stated that Aristotele (BC384 - BC322) considered firstly the division by zero in the sense of physics with many evidences and detailed discussions.

In fact, he stated strongly in the last part of the paper as follows:

> Tradition in this particular may prove to be trustworthy, but it necessarily must be rejected with respect to the more problem. Historical evidence points to Aristotele, rather than Btrahmaguputa, as the one who first considered division by zero.

However, in a strict sense, Brahmagupta (598 - 668 ?) introduced zero and he already defined as $0/0 = 0$ in Brhmasphuasiddhnta (628). However, our world history stated that his definition $0/0 = 0$ is wrong over 1300 years, but, we showed that his definition is suitable. For the details, see the references.

India is great for mathematical sciences and philosophy, because basic arithmetic operations were discovered by Brahmagupta in 628 with zero, negative numbers and so on. However, his basic ideas were derived on the long history of India for void, nothing, infinity, non-existence and existence and so on. For example, in Vedas ([30]), we can find the decimal number system in very old days.

From the recent articles, we can study the related essential history. From [63, 64], we can see the long history of division by zero in India. For the great history of India for mathematics, we can see from [21, 59, 30].

In particular, we can see that Europian countries were very weak on ZERO and arithmetics from, for example, [59].

Typically, F. Cajori ([10]) (1929) stated that Bernard Bdzano stated impossibility of the division by zero by showing a contradiction by the cancellation by zero. Meanwhile, C. W. Dodge ([18]) (1990) showed that from the algebraic viewpoint, the division by zero is impossible.

We will recall the recent articles on the division by zero. J. A. Bergstra, Y. Hirshfeld and J. V. Tucker [7] and J. A. Bergstra [8] discussed the relationship between fields and the division by zero, and the importance of the division by zero for computer science. They, however, seem that the relationship of the division by zero and field structures are abstract.

Meanwhile, Carlström ([11]) introduced the wheel theory;

> wheels are a type of algebra where division is always defined. In particular, division by zero is meaningful. The real numbers can be extended to a wheel, as any commutative ring. The Riemann sphere can also be extended to a wheel by adjoining an element $\perp$, where $0/0 = \perp$. The Riemann sphere is an extension of the complex plane by an element ∞, where $z/0 = \infty$ for any complex

> $z \neq 0$. However, $0/0$ is still undefined on the Riemann sphere, but is defined in its extension to a wheel. The term wheel is introduced by the topological picture $\odot$ of the projective line together with an extra point $\perp = 0/0$.

Similarly, T.S. Reis and J.A.D.W. Anderson ([50, 51]) extend the system of the real numbers by defining division by zero with three infinities $+\infty, -\infty, \Phi$ (Transreal Calculus).

However, we can introduce simply a very natural field containing the division by zero that is a natural extension (modification) of our mathematics, as the Yamada field. For the above axiomatic great theories, it seems that some concrete examples derived from the theories are poor and they are abstract ones.

In connection with the deep problem with physics of the division by zero problem, see J. Czajko [13, 14, 15]. However, we will be able to find many logical confusions in the papers, as we refer to the essence later.

J. P. Barukčić and I. Barukčić ([5]) discussed the relation between the division $0/0$ and special relative theory of Einstein. However it seems that their result is curious with their logics. Their results contradict with ours.

L. C. Paulson stated that I would guess that Isabelle has used this **convention** $1/0 = 0$ since the 1980s and introduced his book [34] referred to this fact. However, in his group the importance of this fact seems to be entirely ignored at this moment as we see from the book. Indeed, he sent his email as follows:

> There are situations when it is natural to define x/0 = 0. For example, if you define division using primitive recursion, in which all functions are total, you will get this identity. There is nothing deep about it.
>
> If you adopt this convention, it turns out that some identities involving division hold unconditionally, such as (x+y)/z = x/z + y/z. Other identities continue to require 0 to be treated separately, such as x/x = 1.
>
> The idea that x/0 = 0 is only a convention. It does not change mathematics in any significant way and it does not lead to contradictions either.
> (2017.07.04.00:22).

See also P. Suppes ([61]) for the interesting viewpoint for the division by zero from the viewpoint of logic, pages 163-166.

For the more recent great works, see E. Jeřábek [24] and B. Santangelo [58]. They state in the abstracts of the papers as follows:

E. Jeřábek [24]:

> For any sufficiently strong theory of arithmetic, the set of Diophantine equations provably unsolvable in the theory is algorithmically undecidable, as a consequence of the MRDP theorem. In contrast, we show decidability of Diophantine equations provably unsolvable in Robinson's arithmetic Q. The argument hinges on an analysis

of a particular class of equations, hitherto unexplored in Diophantine literature. We also axiomatize the universal fragment of Q in the process.

B. Santangelo [58]:

> The purpose of this paper is to emulate the process used in defining and learning about the algebraic structure known as a Field in order to create a new algebraic structure which contains numbers that can be used to define Division By Zero, just as i can be used to define $\sqrt{-1}$.
>
> This method of Division By Zero is different from other previous attempts in that each $\frac{\alpha}{0}$ has a different unique, numerical solution for every possible α, albeit these numerical solutions are not any numbers we have ever seen. To do this, the reader will be introduced to an algebraic structure called an S-Structure and will become familiar with the operations of addition, subtraction, multiplication and division in particular S-Structures. We will build from the ground up in a manner similar to building a Field from the ground up. We first start with general S-Structures and build upon that to S-Rings and eventually S-Fields, just as one begins learning about Fields by first understanding Groups, then moving up to Rings and ultimately to Fields. At each step along the way, we shall prove important properties of each S-Structure and of the operations in each of these S-Structures. By the end, the reader will become familiar with an S-Field, an S-Structure which is an extension of a Field in which we may uniquely define $\alpha/0$ for every non-zero α which belongs to the Field. In fact, each $\frac{\alpha}{0}$ has a different, unique solution for every possible α. Furthermore, this Division By Zero satisfies $\alpha/0 = q$ such that $0 \cdot q = \alpha$, making it a true Division Operation.

Meanwhile, we should refer to up-to-date information:

Riemann Hypothesis Addendum - Breakthrough Kurt Arbenz :
https://www.researchgate.net/publication/272022137 Riemann Hypothesis Addendum - Breakthrough.

Here, we recall Albert Einstein's words on mathematics:

> Blackholes are where God divided by zero. I don't believe in mathematics. George Gamow (1904-1968) Russian-born American nuclear physicist and cosmologist remarked that "it is well known to students of high school algebra" that division by zero is not valid; and Einstein admitted it as **the biggest blunder of his life** (Gamow, G., My World Line (Viking, New York). p 44, 1970).

In the usual sense that division is given by the inverse operation of product, division by zero is impossible and so for long years division by zero was not considered seriously among mathematicians. Therefore, division by zero is interested in physicians as in the above Einstein, Aristotele and many related people, because we have many formulas containing the division by zero; typically, for the Newton formula

$$F = G\frac{m_1 m_2}{r^2},$$

we are interested in the case $r = 0$.

Meanwhile, in computer science, division by zero is a typical problem, because, division by zero leads to computer troubles. We know the famous accident of

on September 21, 1997, a division by zero error in the "Remote Data Base Manager" aboard USS Yorktown (CG-48) brought down all the machines on the network, causing the ship's propulsion system to fail,

however, many people will meet to computer troubles with the division by zero, quite popular ways.

The third group with interest with division by zero exists; they wish to consider why "impossibility of division by zero" and they wish to consider the problem in some seriously. This challenges are still continuing nowdays as we refer to in the belows.

In Education, the problem of division by zero is a typical popular topics.

We have still curious situations and opinions on the division by zero; in particular, the two great challengers Jakub Czajko [14] and Ilija Barukčić [5] on the division by zero in connection with physics stated recently that we do not have the definition of the division 0/0, however $0/0 = 1$. They seem to think that a truth is based on physical objects and is not on our mathematics. In such a case, we will not be able to continue discussions on the division by zero more, because for mathematicians, they will not be able to follow their logics more. However, then we would like to ask for the question that what are the values and contributions of your articles and discussions. We will expect some contributions, of course.

This question will reflect to mathematicians contrary. We stated for the estimation of mathematics in [49] as follows. Mathematics is the collection of relations and, good results are fundamental, beautiful, and give good impacts to human beings. With this estimation, we stated that the Euler formula

$$e^{\pi i} = -1$$

is the best result in mathematics in details in:

No.81, May 2012(pdf 432kb) www.jams.or.jp/kaiho/kaiho-81.pdf

In order to show the importance of our division by zero and division by zero calculus we are requested to show their importance. However, with the results stated in the references, we think the importance of our division by zero was already and definitely stated clearly.

It seems that the long and mysterious confusions for the division by zero were on the **definition**. – Indeed, when we consider the division by zero $a/0$ in the usual sense as the solution of the fundamental equation $0 \cdot z = a$, we have immediately the simple contradiction for $a \neq 0$, however, such cases $0/0$ and $1/0$ may happen, in particular, in mathematical formulas and physical formulas. The typical example is the case of $x = 0$ for the fundamental function $y = 1/x$.

– As we stated in the above, some researchers considered that for the mysterious objects $0/0$ and $1/0$, they considered them as ideal numbers as in the imaginary number i from its great success. However, such an idea will not be good as the number system, as we see simply from the concept of the Yamada field containing the division by zero.

Another important fact was **discontinuity** for the function $y = 1/x$ at the origin. Indeed, by the concept of the Moore-Penrose generalized solution of the fundamental equation $ax = b$, the division by zero was trivial and clear all as $a/0 = 0$ in the general fraction that is defined by the generalized solution of the equation $ax = b$. However, for the strong discontinuity of the function $y = 1/x$ at the origin, we were not able to accept the result $a/0 = 0$ for very long years.

As the number system containing the division by zero, the Yamada field structure is simple and complete. However for the applications of the division by zero to **functions**, we will need the concept of **division by zero calculus** for the sake of uniquely determinations of the results and for other reasons.

2. Tiwari's basic ideas

We can understand Tiwari's basic ideas from the 7 pages paper, precisely.

Since the division by zero $z/0$ is not possible in the usual sense that $z/0 = X$ and $z = 0 \times X$ are the same, we have to consider some definition of the division by zero $z/0$.

His first idea: for the fraction

$$B = \frac{A}{Q},$$

we will consider it as follows: it is from the general form

$$A = B \times Q + R.$$

Therefore, for $Q = 0$, we have

$$A = R,$$

and he considers that the division by zero $z/0$ is zero and the remainder is z. This great idea comes from Mahavira (about 800 - about 870).

For his great idea, we have to refer to the same idea and the exact proof that our colleague Hiroshi Michiwaki had, on our early stage discovery of the division by zero (23 Feb. 2014).

His second idea is follows:

For a value of a function $F(z)$, he considers that

$$F(z) = \lim_{\delta \to 0} \frac{F(z - \delta) + F(z + \delta)}{2};$$

that is, with the mean value. And he obtained the very important results

$$\frac{1}{0} = 0, \quad \tan \frac{\pi}{2} = 0,$$

from the functions $y = 1/x$ and $y = \tan x$, respectively.

Of course, we considered the same way on our initial stage of our discovery of the division by zero.

However, with his idea, we will not be able to derive the important result, for example, for the function

$$f(x) = \frac{1}{x^2},$$

$f(0) = 0$.

Furthermore, in his definition, when do not exist the limits, he will not be able to give the definition.

2.1. Conclusion. Incidentally, when we find his publications, we are writing the Annoucement 549; an answer for the question whether mathematics is innovation (creation) or discovery. There we stated that mathematics is the real existence and not innovation. Mathematics exists independently of our existence, independently of time and energy. We have to say that mathematics was created by God. – Absolute existences. Indeed, we wrote: What is mathematics?

No.81, May 2012(pdf 432kb)
www.jams.or.jp/kaiho/kaiho-81.pdf

in Japanese, in details with human beings.

In particular, mathematics is over logic, we consider so.

From these ideas, we would like to say that the division by zero was discovered by Ankur Tiwari on 2011 based on his 7 pages article at this moment.

One basic reason is that he got the great ideas on the great history of India on the problem:

Brahmagupta (598 - 670), his basic result is $0/0 = 0$ and in general $a/0$ is impossible

Mahavira (about 800 - about 870), his basic result is $100/0 = 100$,

and

Bhaskara II (1114 - 1185), his basic result is $100/0 = 1/0 = \infty$.

The second important reason is on his estimation for the results obtained; he admits the importance of the results in a highly way as we see from the document of 7 pages.

Therefore, we had sent the email to him as follows:

Dear Ankur Tiwari:

Indeed, you are great and your discovery is very important. Since my English ability is poor, I first wrote the attached Announcement 550 for its importance in Japanese.

The main points are:

You are the first man of discovery of the division by zero,

Your passion and high estimation to the discovery are important factors.

I would like to send you; Congratulations!!!

You will be extremely happy with the great discovery.
We thought so.
I would like to write a new version as in

viXra:1903.0184 submitted on 2019-03-10 20:57:02,
Who Did Derive First the Division by Zero 1/0 and the Division by Zero Calculus $\tan(\pi/2) = 0, \log 0 = 0$ as the Outputs of a Computer?

And I would like to add your important discovery in my book in details.

With best regards,
Sincerely yours,

Saburou Saitoh
2020.2.28.05:00

Now we think that any estimation ability is important; based on this idea, for the facts that CSEB and Chhattisgarh Academia gave the high estimation on his discovery we would like to express our great respects to them.

Meanwhile, for example, the division by zero is the generalized inverse - in the sense of **Moore-Penrose generalized inverse** - for the fundamental equation $aX = b$ and the inverse is fundamental and popular for the equation. Therefore, since our initial stage of the division by zero study, we stated repeatedly that the division by zero is trivial and clear all. However, over those 7 years, our world may not be accepted our opinion on its importance. Therefore, we are looking for its importance with many evidences over 1100 items.

In addition, we would like to refer to our paper ([12]) that will contain the division by zero as a very special case.

2.2. Misha Gromov defined that $\frac{0}{0} = 0$. At 2020.2.29.08 : 00, we obtained the email from José Manuel Rodríguez Caballero:

Dear Saitoh,
Look at page 5 of the following paper (0 / 0 = 0)
https://www.ihes.fr/ gromov/wp-content/uploads/
2018/08/structre-serch-entropy-july5-2012.pdf
José M.

Surprisingly enough, in the article ([20]) Misha Gromov **defined** that

$$\frac{0}{0} = 0$$

on June 25, 2013.

2.3. Could Brahmagupta derive the result $1/0 = 0$ from his result $0/0 = 0$? Tiwari considers that the result $1/0 = 0$ is derived from the result $0/0 = 0$ as in

$$\frac{1}{0} = \frac{1}{\frac{0}{0}} = \frac{1 \times 0}{0} = \frac{0}{0} = 0.$$

This curious logic may not be accepted and contrary, we think that Brahmagupta was not, in general, able to consider the division by zero $1/0 = 0$. Look ([44]) for this opinion.

3. W. Hövel's interpretation in integers

W. Hövel gave the pleasant interpretation:

Dividing integer Numbers:

A mother invites kids to dinner. She cooks beans. She has M beans in her pot. Now she wants to share the beans fairly among the kids. Her math is very natural; she can only count. So she goes around the table and always gives the K kids sitting at the table a bean on their plate. She repeats this until all of the beans are distributed. Now it can happen that some children have one bean less than the other. That's unfair! So she gathers the excess beans back into her pot, which will contain m beans after the division. Now everyone is satisfied and you can draw up a balance sheet:

M : number of beans in the mother's pot before division
m : number of beans in the mother's pot after division
K : number of kids
k : number of beans on the kid's plate after division
$M = k \times K + m$

Special case $M < K$:

There are more kids at the table than beans in the pot. To be fair, the mother has to collect all the beans back into their pot. The kids were given nothing to eat.

$$m = M$$

$$k = 0$$

Special case $K = 0$:

There are no kids at the table. After the division procedure, the mother still has $m = M$ beans in her pot, just as in the case of $M < K$ above. She sees no difference between these two cases, the pot is still full. Thus $k = 0$, the kids were given nothing to eat.

This is the famous problem that SABUROU SAITOH solved.

Special case $M \gg 1, K \ll M$:

Many beans were cooked in mother's pot and the kids were given a large number of beans on their plates. The beans look more and more like a bean soup. It looks like continuous. Private note for SABUROU SAITOH by Wolfhard Hövel (2020.10.9.17:10).

4. Division by zero and computers

On February 16, 2019 H. Okumura introduced the surprising news in Research Gate:

> José Manuel Rodríguez Caballero
> Added an answer
> In the proof assistant Isabelle/HOL we have $x/0 = 0$ for each number x. This is advantageous in order to simplify the proofs.
> You can download this proof assistant here:
> **https://isabelle.in.tum.de/**

J.M.R. Caballero kindly showed surprisingly several examples by the system that

$$\tan\frac{\pi}{2} = 0,$$

$$\log 0 = 0,$$

$$\exp\frac{1}{x}(x = 0) = 1,$$

and others.

The relation of Isabelle/HOL and division by zero is unclear at this moment, however, the following document will be interested in:

> Dear Saitoh,
> In Isabelle/HOL, we can define and redefine every function in different ways. So, logarithm of zero depends upon our definition. The best definition is the one which simplify the proofs the most. According to the experts, z/0 = 0 is the best definition for division by zero.
>
> $$\tan(\pi/2) = 0$$
>
> $$\log 0 =$$
>
> is undefined (but we can redefine it as 0)
>
> $$e^0 = 1$$
>
> (but we can redefine it as 0)
>
> $$0^0 = 1$$
>
> (but we can redefine it as 0).
> In the attached file you will find some versions of logarithms and exponentials satisfying different properties. This file can be opened with the software Isabelle/HOL from this webpage:
> https://isabelle.in.tum.de/
> Kind Regards,
> José M.
> (2017.2.17.11:09).

At 2019.3.4.18:04 for my short question, we received:

It is as it was programmed by the HOL team.
Jose M.
On Mar 4, 2019, Saburou Saitoh wrote:
Dear José M.
I have the short question.
For your outputs for the division by zero calculus, for the input, is it some direct or do you need some program???
With best regards, Sincerely yours,
Saburou Saitoh 2019.3.4.18:00

Furthermore, for the presentation at the annual meeting of the Japanese Mathematical Society at the Tokyo Institute of Technology:

March 17, 2019; 9:45-10:00 in Complex Analysis Session, *Horn torus models for the Riemann sphere from the viewpoint of division by zero* with [16],

he kindly sent the message:

> It is nice to know that you will present your result at the Tokyo Institute of Technology. Please remember to mention Isabelle/HOL, which is a software in which x/0 = 0. This software is the result of many years of research and a millions of dollars were invested in it. If x/0 = 0 was false, all these money was for nothing. Right now, there is a team of mathematicians formalizing all the mathematics in Isabelle/HOL, where x/0 = 0 for all x, so this mathematical relation is the future of mathematics. https://www.cl.cam.ac.uk/ lp15/Grants/Alexandria/

Surprisingly enough, he sent his email at 2019.3.30.18:42 as follows:

> Nevertheless, you can use that $x/0 = 0$, following the rules from Isabelle/HOL and you will obtain no contradiction. Indeed, you can check this fact just downloading Isabelle/HOL:
> https://isabelle.in.tum.de/
> and copying the following code
> theory DivByZeroSatoih imports Complex Main

5. Our short history of division by zero

By a **natural extension** of the fractions b/a for any complex numbers a and b, we found the simple and beautiful result, for any complex number b

$$\frac{b}{0} = 0, \tag{5.1}$$

incidentally in [53] by the Tikhonov regularization for the Hadamard product inversions for matrices, and we discussed their properties and gave several physical interpretations on the general fractions in [28] for the case of real numbers. The result is a very special case for general fractional functions in [12].

Sin-Ei Takahasi ([28]) discovered a simple and decisive interpretation (5.1) by analyzing the extensions of fractions and by showing the complete characterization for the property (5.1):

Proposition 5.1. *Let F be a function from $\mathbf{C} \times \mathbf{C}$ to $\mathbf{C}$ satisfying*

$$F(b,a)F(c,d) = F(bc,ad) \quad for\ all \quad a,b,c,d \in \mathbf{C}$$

and

$$F(b,a) = \frac{b}{a}, \quad a,b \in \mathbf{C}, a \neq 0.$$

Then, we obtain $F(b,0) = 0$ for any $b \in \mathbf{C}$.

Note that the proposition is proved simply by 2 or 3 lines. In the long mysterious history of the division by zero, this proposition seems to be decisive.

Indeed, the Takahasi's assumption for the product property should be accepted for any generalization of fraction (division). Without the product property, we will not be able to consider any reasonable fraction (division).

Following the proposition, we should **define**

$$F(b,0) = \frac{b}{0} = 0,$$

and consider, for any complex number b, as (5.1); that is, for the mapping

$$W = \frac{1}{z},$$

the image of $z = 0$ is $W = 0$ (**should be defined from the form**). This fact seems to be a curious one in connection with our well-established popular image for the point at infinity on the Riemann sphere. As the representation of the point at infinity of the Riemann sphere by the zero $z = 0$, we will see some delicate relations between 0 and ∞ which show a strong discontinuity at the point of infinity on the Riemann sphere ([33]). We did not consider any value of the elementary function $W = 1/z$ at the origin $z = 0$, because we did not consider the division by zero $1/0$ in a good way. Many and many people consider its value by the limiting like $+\infty$ and $-\infty$ or the point at infinity as ∞. However, their basic idea comes from **continuity** with the common sense or based on the basic idea of Aristotle. – For the related Greece philosophy, see [65, 66, 67]. However, as the division by zero we will consider its value of the function $W = 1/z$ as zero at $z = 0$. We will see that this new definition is valid widely in mathematics and mathematical sciences, see ([33, 42]) for example. Therefore, the division by zero will give great impacts to calculus, Euclidean geometry, analytic geometry, complex analysis and the theory of differential equations in an undergraduate level and furthermore to our basic ideas for the space and universe.

Meanwhile, the division by zero (5.1) was derived from several independent approaches as in:

1) by the generalization of the fractions by the Tikhonov regularization or by the Moore-Penrose generalized solution to the fundamental equation $az = b$ that leads to the definition of the fraction $z = b/a$,

2) by the intuitive meaning of the fractions (division) by H. Michiwaki,

3) by the unique extension of the fractions by S. Takahasi, as in the above,

4) by the extension of the fundamental function $W = 1/z$ from $\mathbf{C} \setminus \{0\}$ into $\mathbf{C}$ such that $W = 1/z$ is a one to one and onto mapping from $\mathbf{C} \setminus \{0\}$ onto $\mathbf{C} \setminus \{0\}$ and the division by zero $1/0 = 0$ is a one to one and onto mapping extension of the function $W = 1/z$ from $\mathbf{C}$ onto $\mathbf{C}$,

and

5) by considering the values of functions with the mean values of functions.

Furthermore, in ([32]) we gave the results in order to show the reality of the division by zero in our world:

A) a field structure as the number system containing the division by zero — the **Yamada field Y**,

B) by the gradient of the y axis on the (x, y) plane — $\tan \frac{\pi}{2} = 0$,

C) by the reflection $W = 1/\overline{z}$ of $W = z$ with respect to the unit circle with center at the origin on the complex z plane — the reflection point of zero is zero, (The classical result is wrong, see [42]),

and

D) by considering rotation of a right circular cone having some very interesting phenomenon from some practical and physical problem.

In ([29]), we gave beautiful geometrical interpretations of determinants from the viewpoint of the division by zero. Furthermore, in ([33],[42]), we discussed many division by zero properties in the Euclidean plane - however, precisely, our new space is not the Euclidean space. More recently, we see the great impact to Euclidean geometry in connection with Wasan in ([38, 36, 40, 41]).

6. Division by zero calculus

As the number system containing the division by zero, the Yamada field structure is complete. However for applications of the division by zero to **functions**, we will need the concept of division by zero calculus for the sake of uniquely determinations of the results and for other reasons.

The short version of this section was given by [48] in the Proceedings of the International Conference.

Therefore, we will introduce the division by zero calculus: For any Laurent expansion around $z = a$,

$$f(z) = \sum_{n=-\infty}^{-1} C_n (z-a)^n + C_0 + \sum_{n=1}^{\infty} C_n (z-a)^n, \tag{6.1}$$

we obtain the identity, by the division by zero

$$f(a) = C_0. \tag{6.2}$$

Note that here, there is no problem on any convergence of the expansion (6.1) at the point $z = a$, because all the terms $(z - a)^n$ are zero at $z = a$ for $n \neq 0$.

For the correspondence (6.2) for the function $f(z)$, we will call it **the division by zero calculus**. By considering the formal derivatives in (6.1), we can **define** any order derivatives of the function f at the singular point a; that is,

$$f^{(n)}(a) = n! C_n.$$

In order to avoid any logical confusion in the division by zero, we would like to refer to the logical essence:

For the elementary function $W = f(z) = 1/z$, we define $f(0) = 0$ and we will write it by $1/0 = 0$ following the form, apart from the intuitive sense of fraction. With only this new definition, we can develop our mathematics, through the division by zero calculus.

As a logical line for the division by zero, we can also consider as follows:

We define $1/0 = 0$ for the form; this precise meaning is that for the function $W = f(z) = 1/z$, we have $f(0) = 0$ following the form. Then, we can define the division by zero calculus. In particular, from the function $f(x) \equiv 0$ we have $0/0 = 0$.

In order to avoid any logical confusion, we would like to state the essence, repeatedly.

Apart from the motivations above, we define the division by zero calculus by (6.1). With this assumption, we can obtain many new results and new ideas. However, for this assumption we have to check the results obtained whether they are reasonable or not. By this idea, we can avoid any logical problems. – In this point, the division by zero calculus may be considered as a new axiom.

Note that for the function $f(z) = z + \frac{1}{z}$, $f(0) = 0$, however, for the function

$$f(z)^2 = z^2 + 2 + \frac{1}{z^2},$$

we have $f^2(0) = 2$. Of course,

$$f(0) \cdot f(0) = \{f(0)\}^2 = 0.$$

We consider the function

$$y = \frac{e^{cx}}{(c-a)(c-b)}.$$

If $c = a(\neq b)$, then, by the division by zero calculus, we have

$$y = \frac{xe^{ax}}{a-b}.$$

If $c = a = b$, then, by the division by zero calculus, we have

$$y = \frac{x^2 e^{ax}}{2}.$$

These functions have the practical meanings in the ordinary differential equations. See [48].

Furthermore, see [33] for many examples.

7. Triangles and division by zero calculus

In order to see how elementary of the division by zero calculus, we will see the division by zero calculus in triangles as a fundamental object.

We will consider a triangle ABC with side length a, b, c. We have the formula

$$\frac{a^2 + b^2 - c^2}{a^2 - b^2 + c^2} = \frac{\tan B}{\tan C}.$$

If $a^2 + b^2 - c^2 = 0$, then $C = \pi/2$. Then,

$$0 = \frac{\tan B}{\tan \frac{\pi}{2}} = \frac{\tan B}{0}.$$

On the other hand, for the case $a^2 - b^2 + c^2 = 0$, then $B = \pi/2$, and we have

$$\frac{a^2 + b^2 - c^2}{0} = \frac{\tan \frac{\pi}{2}}{\tan C} = 0.$$

Let H be the perpendicular leg of A to the side BC, and let E and M be the mid points of AH and BC, respectively. Let θ be the angle of EMB ($b > c$). Then, we have

$$\frac{1}{\tan \theta} = \frac{1}{\tan C} - \frac{1}{\tan B}.$$

If $B = C$, then $\theta = \pi/2$ and $\tan(\pi/2) = 0$.

8. Broken phenomena of figures by area and volume

The strong discontinuity of the division by zero calculus around the point at infinity will appear as the destruction of various figures. These phenomena may be looked in many situations as the universal one. However, the simplest cases are the disc and sphere (ball) with their radius $1/\kappa$. When $\kappa \to +0$, the areas and volumes of discs and balls tend to $+\infty$, respectively, however, when $\kappa = 0$, they are zero, because they become the half-plane and half-space, respectively. These facts were also looked by analytic geometry. However, the results are clear already from the definition of the division by zero and for the function $y = 1/x$ at $x = 0$.

For a function

$$S(x, y) = a(x^2 + y^2) + 2gx + 2fy + c, \tag{8.1}$$

the radius R of the circle $S(x, y) = 0$ is given by

$$R = \sqrt{\frac{g^2 + f^2 - ac}{a^2}}.$$

If $a = 0$, then the area πR^2 of the disc is zero, by the division by zero calculus. In this case, the circle is a line (degenerated).

The center of the circle (8.1) is given by

$$\left(-\frac{g}{a}, -\frac{f}{a}\right).$$

Therefore, the center of a general line

$$2gx + 2fy + c = 0$$

may be considered as the origin $(0, 0)$, by the division by zero.

9. Parallel lines

We write lines by

$$L_k : a_k x + b_k y + c_k = 0, \quad k = 1, 2.$$

The common point is given by, if $a_1 b_2 - a_2 b_1 \neq 0$; that is, the lines are not parallel

$$\left(\frac{b_1 c_2 - b_2 c_1}{a_1 b_2 - a_2 b_1}, \frac{a_2 c_1 - a_1 c_2}{a_1 b_2 - a_2 b_1} \right).$$

By the division by zero, we can understand that if $a_1 b_2 - a_2 b_1 = 0$, then the common point is always given by

$$(0, 0),$$

even two lines are the same. This fact shows that our new image for the Euclidean space is right.

In particular, note that the concept of parallel lines is very important in the Euclidean plane and non-Euclidean geometry. With our sense, there are no parallel lines and all lines pass the origin. This will be our world on the Euclidean plane. However, this property is not geometrical and has a strong discontinuity. This surprising property may be looked also clearly by the polar representation of a line.

We write a line by the polar coordinate

$$r = \frac{d}{\cos(\theta - \alpha)},$$

where $d = \overline{OH} > 0$ is the distance of the origin O and the line such that OH and the line is orthogonal and H is on the line, α is the angle of the line OH and the positive x axis, and θ is the angle of OP ($P = (r, \theta)$ on the line) from the positive x axis. Then, if $\theta - \alpha = \pi/2$; that is, OP and the line is parallel and P is the point at infinity, then we see that $r = 0$ by the division by zero calculus; the point at infinity is represented by zero and we can consider that the line passes the origin, however, it is in a discontinuous way.

10. Descartes circle theorem

The following theorem was considered by Renè Descartes and is called the Descartes circle theorem with many references.

Theorem 10.1. *For mutually touching four circles γ_i $(i = 1, 2, 3, 4)$ of radii r_i, the following equation holds:*

$$\frac{1}{r_4} = \frac{1}{r_1} + \frac{1}{r_2} + \frac{1}{r_3} \pm 2\sqrt{\frac{1}{r_1 r_2} + \frac{1}{r_2 r_3} + \frac{1}{r_3 r_1}}. \tag{10.1}$$

This result and many variations were well-known in Wasan geometry. In [38] we have shown that the theorem can also be considered in the degenerate cases in which some circles are points or lines [38], where one of the highlights is the case in which $\{\gamma_1, \gamma_2, \gamma_3\}$ consists of two proper circles and a point.

For more examples of applications of the definition of division by zero to Wasan geometry and related topics see [37, 38, 39, 40]. In particular, H. Okumura is developing greately the geometry of circles and lines by applying the division by zero calculus that may be looked as entirely new beautiful mathematics. See his recent papers cited in the reference.

For a general situation of our division by zero calculus, see [56]. Their compact version was published in the book [57] that may be considered as the first book on the division by zero calculus and division by zero also in a reasonable sense.

11. Fundametals of the division by zero calculus

We will list up the fundamentals of the division by zero calculus by items, simply from [57].

(1) **Division by zero calculus for differentiable functions:**

We will give the definition of the division by zero calculus for more general functions over analytic functions.

For a function $y = f(x)$ which is n order differentiable at $x = a$, we will **define** the value of the function, for $n > 0$

$$\frac{f(x)}{(x-a)^n}$$

at the point $x = a$ by the value

$$\frac{f^{(n)}(a)}{n!}.$$

For the important case of $n = 1$,

$$\frac{f(x)}{x-a}|_{x=a} = f'(a). \tag{11.1}$$

In particular, the values of the functions $y = 1/x$ and $y = 0/x$ at the origin $x = 0$ are zero. **We write them as** $1/0 = 0$ **and** $0/0 = 0$**, respectively.** Of course, the definitions of $1/0 = 0$ and $0/0 = 0$ are not usual ones in the sense: $0 \cdot x = b$ and $x = b/0$. Our division by zero is given in this sense and is not given by the usual sense.

We will give its naturality of the definition.

Indeed, we consider the function $F(x) = f(x) - f(a)$ and by the definition, we have

$$\frac{F(x)}{x-a}|_{x=a} = F'(a) = f'(a).$$

Meanwhile, by the definition, we have

$$\lim_{x \to a} \frac{F(x)}{x-a} = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = f'(a).$$

The identity (11.1) may be regarded as an interpretation of the differential coefficient $f'(a)$ by the concept of the division by zero. Here, we do not use the concept of limitings. This means that NOT

$$\lim_{x \to a} \frac{f(x)}{x-a}$$

BUT

$$\frac{f(x)}{x-a}|_{x=a}.$$

Note that $f'(a)$ represents the principal variation of order $x-a$ of the function $f(x)$ at $x=a$ which is defined independently of the value $f(a)$. This is a basic meaning of the division by zero calculus $\frac{f(x)}{x-a}|_{x=a}$.

Following this idea, we can accept the formula, naturally, for also $n=0$ for the general formula.

In the expression (11.1), the value $f'(a)$ in the right hand side is represented by the point a, meanwhile the expression

$$\frac{f(x)}{x-a}|_{x=a}$$

in the left hand side, is represented by the dummy variable $x-a$ that represents the property of the function around the point $x=a$ with the sense of the division

$$\frac{f(x)}{x-a}.$$

For $x \neq a$, it represents the usual division.

Of course, by our definition

$$\frac{f(x)}{x-a}|_{x=a} = \frac{f(x)-f(a)}{x-a}|_{x=a},$$

however, here $f(a)$ may be replaced by any constant. This fact looks like showing that the function

$$\frac{1}{x-a}$$

is zero at $x=a$. Of course, this result is directly derived from the definition.

When we apply the relation (11.1) in the elementary formulas for differentiable functions, we can imagine some deep results. For example, from the simple formula

$$(u+v)' = u' + v',$$

we have the result

$$\frac{u(x)+v(x)}{x-a}|_{x=a} = \frac{u(x)}{x-a}|_{x=a} + \frac{v(x)}{x-a}|_{x=a},$$

that is not trivial in our definition.

In the following well-known formulas, we have some deep meanings on the division by zero calculus.

$$(uv)' = u'v + uv',$$

$$\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$$

and the famous laws

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt}$$

and

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1.$$

Note also the logarithm derivative, for $u, v > 0$

$$(\log(uv))' = \frac{u\prime}{u} + \frac{v\prime}{v}$$

and for $u > 0$

$$(u^v)' = u^v \left(v' \log u + v\frac{u'}{u} \right).$$

(2) **From Riemann sphere to Däumler - Puha horn torus; or from Euclid - Riemann to Däumler - Puha horn torus model:**

V. V. Puha discovered the mapping of the extended complex plane to a beautiful horn torus at (2018.6.4.7:22) and its inverse at (2018.6.18.22:18).

Incidentally, independently of the division by zero, W. W. Däumler has various special great ideas on horn torus as we see from his site:

Horn Torus & Physics (https://www.horntorus.com/) Geometry Of Everything, intellectual game to reveal engrams of dimensional thinking and proposal for a different approach to physical questions ...

Indeed, W. W. Däumler was presumably the first (1996) who came to the idea of the possibility of a mapping of extended complex plane onto the horn torus. He expressed this idea on his private website:
(http://www.dorntorus.de).

He was also, apparently, the first to point out that zero and infinity are represented by zero and the same point on the horn torus model of extended complex plane.

W. W. Däumler discovered a surprising conformal mapping from the extended complex plane to the horn torus model
(2018.8.18.09):

https://www.horntorus.com/manifolds/conformal.html
and
https://www.horntorus.com/manifolds/solution.html

Absolute function theory.

We will discuss on Däumler's horn torus model from some fundamental viewpoints.

First of all, note that in the Puha mapping and the Däumler mapping, and even in the classical stereographic mapping, we find the division by zero $1/0 = 0/0 = 0$.

What is the number system?

What are the numbers? What is the number system? For these fundamental questions, we can say that the numbers are complex numbers

C and the number system is given by the Yamada field with the simple structure **as a field containing the division by zero.**

Nowadays, we have still many opinions on these fundamental questions, however, this subsection excludes all those opinions as in the above.

What is the natural coordinates?

We represented the complex numbers **C** by the complex plane or by the points on the Riemann sphere. On the complex plane, the point at infinity is the ideal point and for the Riemann sphere representation, we have to accept the **strong discontinuity**. From these reasons, the numbers and the numbers system should be represented by the Däumler's horn torus model that is conformally equaivalent to the extended complex plane.

What is a function?, and what is the graph of a function?

A function may be considered as a mapping from a set of numbers into a set of numbers.

The numbers are represented by Däumler's horn torus model and so, we can consider that a function, in particular, an analytic function can be considered as a mapping from Däumler's horn torus model into (or onto) Däumler's horn torus model.

Absolute function theory.

Following the above considerings, for analytic functions when we consider them as the mappings from

Däumler's horn torus model into (or onto) Däumler's horn torus model,

we would like to say that it is an **absolute function theory.**

For the classical theory of analytic functions, discontinuity of functions at singular points will be the serious problems and the theory will be quite different from the new mathematics, when we consider the functions on the Däumler's horn torus model. Even for analytic function theory on bounded domains, when we consider their images on Däumler's horn torus model, the results will be very interesting.

New mathematics and future mathematicians.

The structure of Däumler's horn torus model is very involved and so, we will need some computer systems like MATHEMATICA and Isabelle/HOL system for our research activity. Indeed, for the analytical proof of the conformal mapping of Däumler, we had to use MATHEMATICA, already. Here, we will be able see some future of mathematicans.

For the properties of horn torus with physical applications, see [6].

See also the site of Däumler for some deep ideas:

https://www.horntorus.com/rotations.html

(3) **Derivative of a function:**

On differential coefficients (derivatives), we obtain new concepts, from the division by zero calculus. At first, we will consider the fundamental properties. From the viewpoint of the division by zero, when there exists the limit, at x

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \infty \tag{11.2}$$

or

$$f'(x) = -\infty, \tag{11.3}$$

both cases, we can write them as follows:

$$f'(x) = 0. \tag{11.4}$$

This definition is reasonable, because the point at infinity is represented by 0.

This property was also derived from the fact that the gradient of the y axis is zero; that is,

$$\tan \frac{\pi}{2} = 0, \tag{11.5}$$

that was looked from many geometric properties in [33], and also in the formal way from the result $1/0 = 0$. Of course, by the division by zero calculus, we can derive analytically the result, because

$$\tan x = -\frac{1}{x - \pi/2} + \frac{1}{3}(x - \pi/2) + \frac{1}{45}(x - \pi/2)^3 + \cdots .$$

From the reflection formula of the Psi (Digamma) function

$$\psi(1-z) = \psi(z) + \pi \frac{1}{\tan \pi z}$$

([1], 258), we have, for $z = 1/2$,

$$\tan \frac{\pi}{2} = 0.$$

Note also from the identity

$$\frac{1}{\psi(1-z) - \psi(z)} = \frac{\tan \pi z}{\pi},$$

we have

$$\frac{1}{\psi(1-z) - \psi(z)}(z = 0) = 0$$

and

$$\frac{1}{\psi(1-z) - \psi(z)}\left(z = \frac{\pi}{2}\right) = 0.$$

We will look this fundamental result by elementary functions. For the function

$$y = \sqrt{1 - x^2},$$
$$y' = \frac{-x}{\sqrt{1 - x^2}},$$

and so,

$$[y']_{x=1} = 0, \quad [y']_{x=-1} = 0.$$

Of course, depending on the context, we should refer to the derivatives of a function at a point from the right hand direction and the left hand direction.

Here, note that, for $x = \cos\theta, y = \sin\theta$,

$$\frac{dy}{dx} = \frac{dy}{d\theta}\left(\frac{dx}{d\theta}\right)^{-1} = -\cot\theta.$$

Note also that from the expansion

$$\cot z = \frac{1}{z} + \sum_{\nu=-\infty,\nu\neq 0}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right) \tag{11.6}$$

or the Laurent expansion

$$\cot z = \sum_{n=-\infty}^{\infty} \frac{(-1)^n 2^{2n} B_{2n}}{(2n)!} z^{2n-1},$$

we have

$$\cot 0 = 0.$$

Note that in (11.6), since

$$\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right)_{\nu=0} = \frac{1}{z},$$

we can write it simply

$$\cot z = \sum_{\nu=-\infty}^{+\infty}\left(\frac{1}{z-\nu\pi} + \frac{1}{\nu\pi}\right).$$

We note that in many and many formulas we can apply this convention and modification.

The differential equation

$$y' = -\frac{x}{y}$$

with a general solution

$$x^2 + y^2 = a^2$$

is satisfied for all points of the solutions by the division by zero. However, the differential equations

$$x + yy' = 0, \quad y' \cdot \frac{y}{x} = -1$$

are not satisfied for the points $(-a, 0)$ and $(a, 0)$.

In many and many textbooks, we find the differential equations, however, they are not good in this viewpoint.

For the function $y = \log x$,

$$y' = \frac{1}{x}, \tag{11.7}$$

and so,

$$[y']_{x=0} = 0. \tag{11.8}$$

For the elementary ordinary differential equation

$$y' = \frac{dy}{dx} = \frac{1}{x}, \quad x > 0, \tag{11.9}$$

how will be the case at the point $x = 0$? From its general solution, with a general constant C

$$y = \log x + C, \tag{11.10}$$

we see that

$$y'(0) = \left[\frac{1}{x}\right]_{x=0} = 0, \tag{11.11}$$

that will mean that the division by zero $1/0 = 0$ is very natural.

In addition, note that the function $y = \log x$ has infinite order derivatives and all values are zero at the origin, in the sense of the division by zero calculus.

However, for the derivative of the function $y = \log x$, we have to fix the sense at the origin, clearly, because the function is not differentiable in the usual sense, but it has a singularity at the origin. For $x > 0$, there is no problem for (11.9) and (11.10). At $x = 0$, we see that we can not consider the limit in the usual sense. However, for $x > 0$ we have (11.10) and

$$\lim_{x \to +0} (\log x)' = +\infty. \tag{11.12}$$

In the usual sense, the limit is $+\infty$, but in the present case, in the sense of the division by zero, we have the identity

$$\left[(\log x)'\right]_{x=0} = 0$$

and we will be able to understand its sense graphically.

Note that the function

$$y = ax + b + \frac{1}{x}$$

and its derivative

$$y' = a - \frac{1}{x^2}.$$

Then, the tangential approximate line at $x = 0$ of the function is the y axis and so the gradient of the function at the origin may be considered as zero, however, the derivative at the origin in our sense at the singular point is a.

However, note that the gradients of the tangential lines of the curve converge to a when x tends to $+\infty$, and the origin and the point at infinity are coincident; that is the curve has two tangential lines at the origin (at the point at infinity) and their gradients are zero and a.

By the new interpretation for the derivative, we can arrange the formulas for derivatives, by the division by zero. The formula

$$\frac{dx}{dy} = \left(\frac{dy}{dx}\right)^{-1} \tag{11.13}$$

is very fundamental. Here, we considered it for a local one to one correspondence of the function $y = f(x)$ and for nonvanishing of the denominator

$$\frac{dy}{dx} \neq 0. \tag{11.14}$$

However, if a local one to one correspondence of the function $y = f(x)$ is ensured like the function $y = x^3$ around the origin, we do not need the assumption (11.14). Then, for the point $dy/dx = 0$, we have, by the division by zero,

$$\frac{dx}{dy} = 0.$$

This will mean that the function $x = g(y)$ has the zero derivative and the tangential line at the point is a parallel line to the y- axis. In this sense the formula (11.13) is valid, even the case $dy/dx = 0$. The nonvanising case, of course, the identity

$$\frac{dy}{dx} \cdot \frac{dx}{dy} = 1 \tag{11.15}$$

holds. When we put the vanishing case, here, we obtain the identity

$$0 \times 0 = 1, \tag{11.16}$$

in a sense. Of course, it is not valid, because (11.15) is unclear for the vanishing case. Such an interesting property was referred to by M. Yamane in ([28]).

Furthermore, the formulas

$$\left(\frac{1}{f}\right)' = -\frac{f'}{f^2},$$

$$\left(\frac{1}{f}\right)'' = \frac{2(f')^2 - ff''}{f^3},$$

$$\left(\frac{1}{f}\right)''' = \frac{6ff'f'' - 6(f')^3 - f^2f'''}{f^4},$$

..., and so on, are valid, even the case

$$f(x_0) = 0,$$

at the point x_0.

In those identities in the framework of analytic functions, at first we consider their formulas except singular points and then, following the definition of division by zero calculus at singular points we consider the valid identities. For the case of functions that are not analytic functions, we have to consider case by case at singular points by division by zero or division by zero calculus idea and we have to check the results.

The derivative of the function

$$f(x) = \sqrt{x}(\sqrt{x} + 1)$$

$$f'(x) = \frac{1}{2\sqrt{x}}(\sqrt{x} + 1) + \sqrt{x} \cdot \frac{1}{2\sqrt{x}}$$
$$= \frac{1}{2\sqrt{x}} + \frac{\sqrt{x}}{\sqrt{x}}$$

is valid at even the origin by using the function $\frac{\sqrt{x}}{\sqrt{x}}$ (V. V. Puha: 2018. June). He derived such formulas by using the function x/x.

In particular, note that the division by zero calculus is not almighty. The notation

$$\Delta(x) = \frac{x}{x} = x \cdot \frac{1}{x} = \begin{cases} 0 & \text{for } x = 0 \\ 1 & \text{for } x \neq 0 \end{cases}$$

will be convenient in connection with the Dirac delta function $\delta(x)$.

Implicit functions.

In the function $y = y(x)$ defined by a differentiable implicit function $f(x, y) = 0$, we have the formula

$$\frac{dy}{dx} = -\frac{f_x(x, y)}{f_y(x, y)}.$$

If $f_y(a, b) = 0$, then the tangential line through the point (a, b) of the function is given by

$$f_x(a, b)(x - a) + f_y(a, b)(y - b) = 0,$$

that is $x = a$. Then we have

$$\frac{dy}{dx}(a, b) = -\frac{f_x(a, b)}{0} = 0.$$

Differential quotients and division by zero.

We will refer to an interesting interpretation of the relation between differential and division by zero.

For the differential quotient

$$\frac{dy}{dx},$$

if it is zero in some interval, then, of course, we have that $y = C$ in the interval with a constant C. This will mean that if $dy = 0$, then $y = C$ in some interval with a constant C and $y' = 0$.

Meanwhile, if $dx = 0$, then, by the division by zero, we have

$$\frac{dy}{dx} = 0$$

and so, we have that $y' = 0$. Then, however, $x = D$ with a constant D in some y interval. This interpretation shows that the gradient of the y axis is zero, that is

$$\tan\frac{\pi}{2} = 0.$$

(4) **Introduction to** $\log 0 = \log\infty = 0$:

For any fixed complex number a, we will consider the sector domain $\Delta_a(\alpha,\beta)$ defined by

$$0 \leq \alpha < \arg(z-a) < \beta < 2\pi$$

on the complex z plane and we consider the conformal mapping of $\Delta_a(\alpha,\beta)$ into the complex W plane by the mapping

$$W = \log(z-a).$$

Then, the image domain is represented by

$$S(\alpha,\beta) = \{W; \alpha < \Im W < \beta\}.$$

Two lines $\{W; \Im W = \alpha\}$ and $\{W; \Im W = \beta\}$ usually were considered as having the common point at infinity, however, in the division by zero, the point is represented by zero.

Therefore, $\log 0$ and $\log\infty$ **should be defined as zero**. Here, $\log\infty$ is precisely given in the sense of $[\log z]_{z=\infty}$. However, the properties of the logarithmic function should not be expected more, we should consider the value only. For example,

$$\log 0 = \log(2\cdot 0) = \log 2 + \log 0$$

is not valid.

In particular, in many formulas in physics, in some expression, for some constants A, B

$$\log\frac{A}{B}, \tag{11.17}$$

if we consider the case that A or B is zero, then we should consider it in the form

$$\log\frac{A}{B} = \log A - \log B, \tag{11.18}$$

and we should put zero in A or B. Then, in many formulas, we will be able to consider the case that A or B is zero. For the case that A or B is zero, the identity (11.17) is not valid, then the expression $\log A - \log B$ may be valid in many physical formulas. However, the results are case by case, and **we should check the obtained results for applying the formula** (11.18) **for** $A = 0$ **or** $B = 0$**.** Then, we will be able to enjoy the formula apart from any logical problems as in the applications of the division by zero and division by zero calculus.

From the theory of the hyperfunction theory, we have

$$\lim_{N\to\infty}\frac{1}{2\pi i}\log(N+1-z) = 0$$

in the natural sense ([25], page 25).

Applications of $\log 0 = 0$.

We can apply the result $\log 0 = 0$ for many cases as in the following way.

For example, we will consider the differential equation

$$y = xy' - \log y'.$$

We have the general solution

$$y = Cx - \log C$$

and the singular solution

$$y = 1 + \log x.$$

For $C = 0$, we have $y = 0$, by the division by zero, that is a reasonable solution.

For the differential equation

$$y' = 1 + \frac{y}{x},$$

we have the general solution

$$y = x(\log x + C).$$

How will be at $x = 0$? From

$$y' = \log x + C + 1$$

and

$$y'(0) = C + 1,$$

we have, for $x = 0$

$$\frac{y}{x} = C$$

and so, we see that for $x = 0$, the differential equation is satisfied.

For the differential equation

$$y' + \frac{1}{x}y = y^2 \log x,$$

we have the general solution

$$xy\{C - (\log x)^2\} = 2.$$

Dividing by C and by setting $C = 0$, by the division by zero, we have also the solutions $x = 0$ and $y = 0$.

In the differential equation

$$x^2 y''' + 4x^2 y'' - 2xy' - 4y = \log x,$$

we have the general solution

$$y = \frac{C_1}{x} + \frac{C_2}{x^2} + C_3 x^2 - \frac{1}{4}\log x + \frac{1}{4},$$

satisfying that at the origin $x = 0$

$$y(0) = \frac{1}{4}, y'(0) = 0, y''(0) = 2C_3, y'''(0) = 0.$$

We can give the values C_1 and C_2. For the sake of the division by zero, we can, in general, consider differential equations even at analytic and isolated singular points.

For the formula

$$\int \frac{dx}{\sqrt{x^2 - a^2}} = \cosh^{-1} \frac{x}{a} \quad (x > a > 0)$$

and

$$\cosh^{-1} z = \log(2z) - \frac{1}{4z^2} + ... \quad |z| > 1,$$

we have, for $a = 0$,

$$\int \frac{dx}{x} = \log 2 + \log x.$$

However, here in

$$\log 2 + \log x - \log a,$$

we have to have $\log 0 = 0$.

O. Ufuoma introduced the example at (2019.12.28.5:39):
We consider the equation

$$2^x = 0,$$

then, from $x \log 2 = \log 0 = 0$, we have $x = 0$. Meanwhile, if

$$2^{(x-a)} = 0,$$

we have $x = a$. However, if $a \neq 0$, and if

$$2^{(x-a)} = 2^x 2^{-a},$$

then we have $x = 0$, a contradiction. Therefore, the identity is not valid in this case.

Anyhow, $\log 0 = 0$ is defined by a special sense and so, the derived results should be checked, case by case.

A finite part of divergence integrals.

For a finite part of divergence integrals,

$$\log_{\varepsilon \to 0} \log \varepsilon = 0$$

is very natural and convenient. See [26], pages161-162.

Robin constant and Green's functions.

From the typical case, we will consider a fundamental application. Let $D(a, R) = \{|z| > R\}$ be the outer disc on the complex plane. Then, the Riemann mapping function that maps conformally onto the unit disc $\{|W| < 1\}$ and the point at infinity to the origin is given by

$$W = \frac{R}{z - a}.$$

Therefore, the Green function $G(z, \infty)$ of $D(a, R)$ is given by

$$G(z, \infty) = -\log \left\{ \frac{R}{|z - a|} \right\}.$$

Therefore, from the representation

$$G(z, \infty) = -\log R + \log |z| + \log \left(1 - \frac{a}{|z|} \right),$$

we have the identity

$$G(\infty, \infty) = -\log R,$$

that is the Robin constant of $D(a, R)$. This formula is valid in the general situation, because the Robin constant is defined by

$$\lim_{z \to b} \{G(z, b) + \log |z - b|\},$$

for a general Green function with pole at b of some domain ([2]).

Division by zero calculus for harmonic functions.

For a harmonic function $h(z, a)$ with an isolated singular point at $z = a$ around $z = a$, we consider the analytic function

$$f(z, a) = A \log(z - a) + \sum_{n=-\infty}^{\infty} C_n (z - a)^n; \quad 0 < r < |z - a| < R,$$

whose real part is $h(z, a)$ with constants A and C_n.

Then, we **define** the division by zero calculus for the function $h(z, a)$ at $z = a$ by

$$h(a, a) = \Re C_0.$$

For example, for the Neumann function on the disc $|z| < R$ with the pole at $z = a$

$$N(z, a) = \log \frac{R^3}{|z - a||R^2 - \overline{a}z|},$$

we have

$$N(a, a) = \log \frac{R^3}{|R^2 - |a|^2|}.$$

For the famous Robin constant, this value seems not to be considered.

$e^0 = 1, 0$.

By the introduction of the value $\log 0 = 0$, as the inversion function $y = e^x$ of the logarithmic function, we will consider that $y = e^0 = 0$. Indeed, we will show that this definition is very natural.

We will consider the conformal mapping $W = e^z$ of the strip

$$S(-\pi i, \pi i) = \{z; -\pi < \Im z < \pi\}$$

onto the whole W plane cut by the negative real line $(-\infty, 0]$. Of course, the origin 0 corresponds to 1. Meanwhile, we see that the negative line $(-\infty, 0]$ corresponds to the negative real line $(-\infty, 0]$. In particular, on the real line $\lim_{x\to-\infty} e^x = 0$. In our new space idea from the division by zero, the point at infinity is represented by zero and therefore, we should define as

$$e^0 = 0.$$

For the fundamental exponential function $W = \exp z$, at the origin, we should consider 2 values. The value 1 is the natural value as a regular point of the analytic function, meanwhile the value 0 is given with a strong discontinuity; however, this value will appear in the universe in a natural way.

For the elementary functions $y = x^n, n = \pm 1, \pm 2, \cdots$, we have

$$y = e^{n \log x}.$$

Then, we wish to have

$$y(0) = e^{n \log 0} = e^0 = 0.$$

As a typical example, we will consider the simple differential equation

$$\frac{dx}{x} - \frac{2ydy}{1+y^2} = 0.$$

Then, by the usual method,

$$\log|x| - \log|1+y^2| = C;$$

that is,

$$\log\left|\frac{x}{1+y^2}\right| = \log e^C = \log K, K = e^C > 0$$

and

$$\frac{x}{1+y^2} = \pm K.$$

However, the constant K may be taken as zero, as we see directly $\log e^C = \log K = 0$.

Meanwhile, we will consider the Fourier integral

$$\int_{-\infty}^{\infty} e^{-i\omega t} e^{-\alpha|t|} dt = \frac{2\alpha}{\alpha^2 + \omega^2}.$$

For the case $\alpha = 0$, if this formula valid, then we have to consider $e^0 = 0$.

Furthermore, by Poisson's formula, we have

$$\sum_{n=-\infty}^{\infty} e^{-\alpha|n|} = \sum_{n=-\infty}^{\infty} \frac{2\alpha}{\alpha^2 + (2\pi n)^2}.$$

If $e^0 = 0$, then the above identity is still valid, however, for $e^0 = 1$, the identity is not valid. We have many examples.

For the integral

$$\int_0^\infty \frac{x^3 \sin(ax)}{x^4+4} dx = \frac{\pi}{2} e^{-a} \cos a,$$

the formula is valid for $a = 0$.

For the integral

$$\int_0^\infty \frac{\xi \sin(x\xi)}{1+a^2\xi} d\xi = \frac{\pi}{2a^2} e^{-(x/a)}, \quad x > 0,$$

the formula is valid for $x = 0$.

For the identity

$$x^p + y^p = z^p,$$

for $p = 0$, we would like to consider $e^0 = 0$ from $x^p = \exp(p \log x)$.

Here, in particular, consider the cases: $p = 1/2$ and $x = 0$. Then, we have the natural result

$$0^{1/2} = \sqrt{0} = 0.$$

$0^0 = 1, 0$.

By the standard definition, we will consider

$$0^0 = \exp(0 \log 0) = \exp 0 = 1, 0.$$

The value 1 is famous which was derived by N. Abel, meanwhile, H. Michiwaki had directly derived it as 0 from the result of the division by zero. However, we now know that $0^0 = 1, 0$ is the natural result.

We will see its reality.

For $0^0 = 1$**:**

In general, for $z \neq 0$, from $z^0 = e^{0 \log z}$, $z^0 = 1$, and so, we will consider that $0^0 = 1$ in a natural way.

For example, in the elementary expansion

$$(1+z)^n = \sum_{k=0}^{n} {}_nC_k z^k$$

the formula $0^0 = 1$ will be convenient for $k = 0$ and $z = 0$.

In the fundamental definition

$$\exp z = \sum_{k=0}^{\infty} \frac{1}{k!} z^k$$

in order to have a sense of the expansion at $z = 0$ and $k = 0$, we have to accept the formula $0^0 = 1$.

In the differential formula

$$\frac{d^n}{dx^n} x^n = n x^{n-1},$$

in the case $n = 1$ and $x = 0$, the formula $0^0 = 1$ is convenient and natural.

In the Laurent expansion, if $0^0 = 1$, it may be written simply as

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z-a)^n,$$

for $f(a) = C_0$.

For k: $k^2 < 1$, we have the identities

$$\sum_{n=0}^{\infty} k^n \sin(n+1)\theta = \frac{\sin\theta}{1 - 2k\cos\theta + k^2}$$

and

$$\sum_{n=0}^{\infty} k^n \cos(n+1)\theta = \frac{\cos\theta - k}{1 - 2k\cos\theta + k^2}.$$

In those identities, for $k = 0$, we have to have $0^0 = 1$.

For $0^0 = 0$:

For any positive integer n, since $z^n = 0$ for $z = 0$, we wish to consider that $0^0 = 0$ for $n = 0$.

For the expansion

$$\frac{t}{\exp t - 1} = \sum_{n=0}^{\infty} \frac{B_n}{n!} t^n,$$

with the Bernoulli's constants B_n, the usual value of the function at $t = 0$ is 1 and this meets the value $0^0 = 1$. Meanwhile, by the division by zero, we have the value 0 by the method

$$\frac{t}{\exp t - 1}|_{t=0} = \frac{0}{\exp 0 - 1} = \frac{0}{0} = 0$$

and this meets with $0^0 = 0$. Note that by the division by zero calculus, we have the value 0 (V. V. Puha: 2018.7.3.6:01).

Philip Lloyed's question (2019.1.18): *What is the value of the equation*

$$x^x = x$$

?

By the equation

$$x(x^{x-1} - 1) = 0,$$

we have $x = 0$ and $x = 1$ therefore, we have also $0^0 = 0$.

P. Lloyed discovered also the solution -1, as we see the result directly and interestingly.

Khandakar Kawkabum Munir Saad asked the question for the equation $2^x = 0$ in Quora: 2019.7.4.17:00. We can give the solution $x = 0$. Therefore, for the very interesting equation $x^2 = 2^x$ we have the trivial solutions 2 and 4 and furthermore, the solution 0.

The values $e^0 = 0$ may be considered that the values at the point at infinity are reflected to the origin and other many functions will have the same property.

(5) **What is the Zero?:**

The zero 0 as the complex number or real number is given clearly by the axioms by the complex number field and real number field, respectively.

For this fundamental idea, we should consider the **Yamada field** containing the division by zero. The Yamada field and the division by zero calculus will arrange our mathematics, beautifully and completely; this will be our real and complete mathematics.

Standard value.

The zero is a center and stand point (or bases, a standard value) of the coordinates - here we will consider our situation on the complex or real 2 dimensional spaces. By the stereographic projection mapping or the Yamada field, the point at infinity 1/0 is represented by zero. The origin of the coordinates and the point at infinity correspond with each other.

As the standard value, for the point $\omega_n = \exp\left(\frac{\pi}{n}i\right)$ on the unit circle $|z| = 1$, for $n = 0$,

$$\omega_0 = \exp\left(\frac{\pi}{0}i\right) = 1, \quad \frac{\pi}{0} = 0.$$

For the mean value

$$M_n = \frac{x_1 + x_2 + ... + x_n}{n},$$

we have

$$M_0 = 0 = \frac{0}{0}.$$

Fruitful world.

For example, in very general partial differential equations, if the coefficients or terms are zero, we have some simple differential equations and the extreme case is all terms zero; that is, we have the trivial equation $0 = 0$; then its solution is zero. When we consider the converse, we see that the zero world is a fruitful one and it means some vanishing world. Recall the Yamane phenomena, the vanishing result is very simple zero, however, it is the result from some fruitful world. Sometimes, zero means void or nothing world, however, it will show some change as in the Yamane phenomena.

From 0 **to** 0**;** 0 **means all and all are** 0.

As an interesting figure (*our life figure*) which shows an interesting relation between 0 and infinity, we will consider a sector Δ_α on the complex $z = x + iy$ plane

$$\Delta_\alpha = \left\{|\arg z| < \alpha; 0 < \alpha < \frac{\pi}{2}\right\}.$$

We will consider a disc inscribed in the sector Δ_α whose center $(k, 0)$ with its radius r. Then, we have

$$r = k \sin \alpha.$$

Then, note that as k tends to zero, r tends to zero, meanwhile k tends to $+\infty$, r tends to $+\infty$. However, by our division by zero calculus, we see that immediately

$$[r]_{r=\infty} = 0.$$

On the sector, we see that from the origin as the point 0, the inscribed discs are increasing endlessly, however their final disc reduces to the origin suddenly - it seems that the whole process looks like our life in the viewpoint of our initial and final.

This will show some great relation between zero and infinity. We can see some mysterious property around the point at infinity.

On the horn torus models of Puha and Däumler, the result is clear. As we see from our life figure, a story starts from the zero and ends to the zero. This will mean that 0 means all and all are 0, in a sense. The zero is a mother of all.

Note that all the equations are stated as equal zero; that will mean that all are represented by zero in a sense.

Impossibility.

As the solution of the simplest equation

$$ax = b \tag{11.19}$$

we have $x = 0$ for $a = 0, b \neq 0$ as the standard value, or the Moore-Penrose generalized inverse. This will mean in a sense, the solution does not exist; to solve the equation (11.19) is impossible. We saw for different parallel lines or different parallel planes, their common point is the origin. Certainly they have the common point of the point at infinity and the point at infinity is represented by zero. However, we can understand also that they have no solutions, no common points, because the point at infinity is an ideal point.

We will consider the point P at the origin with starting at the time $t = 0$ with velocity $V > 0$ and the point Q at the point $d > 0$ with velocity $v > 0$. Then, the time of coincidence P=Q is given by

$$T = \frac{d}{V - v}.$$

When $V = v$, we have, by the division by zero, $T = 0$. This zero represents impossibility. We have many such situations.

We will consider the simple differential equation

$$m\frac{d^2x}{dt^2} = 0, m\frac{d^2y}{dt^2} = -mg \tag{11.20}$$

with the initial conditions, at $t = 0$

$$\frac{dx}{dt} = v_0 \cos\alpha, \quad \frac{dy}{dt} = v_0 \sin\alpha; \quad x = y = 0.$$

Then, the highest high h, arriving time t, the distance d from the starting point at the origin to the point $y(2t) = 0$ are given by

$$h = \frac{v_0^2 \sin\alpha}{2g}, \quad d = \frac{v_0^2 \sin 2\alpha}{g}$$

and

$$t = \frac{v_0 \sin\alpha}{g}.$$

For the case $g = 0$, we have $h = d = t = 0$. We considered the case that they are infinity; however, our mathematics means zero, which shows impossibility.

These phenomena were looked in many cases on the universe; it seems that God does not like the infinity.

As we stated already in the Bhāskara's example – sun and shadow

Zero represents void or nothing.

On ZERO, the authors S. K. Sen and R. P. Agarwal [59] published its history and many important properties. See also R. Kaplan [27] and E. Sondheimer and A. Rogerson [60] on the very interesting books on zero and infinity.

India has a great tradition on ZERO, VOID and INFINITY and they are familiar with those concepts.

Meanwhile, Europian (containing the USA) people do not like such basic ideas and they are not familiar with them.

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