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# RISK PREMIUM AND MERTON FRACTION FOR REGRESSIVE ASSET

AUGUSTE MUHAU AND JULIUS N. ESUNGE

Abstract. The relationship between the risk premium, the asset trend (-risk) and the Merton fraction is close. In this paper, we define amd discuss the concept of regressive assets and show the expression of the risk premium when the asset is risky in the sense that, in addition to the perturbations, it tends to regress. By applying dynamic programming to this asset and optimal control, we determine the Merton fraction for this regressive asset. We present a fact on the relation between the risk premium and the Merton fraction of a regressive asset and demonstrate the proposition with a numerical simulation.

## 1. Introduction

In general, except at the start time, the price of an asset with risk over time is random, hence it is difficult to predict. However, probability theory allows us to better model this random trajectory linked to several types of perturbations. It is therefore becoming more beneficial for investors to know with sufficient certainty the price dynamics of a risky asset in order to take an optimal strategy [12]. For more than forty years, several models predicting economic behavior have been modified to respond to the challenge of fairly high volatility. As a result, much attention has been paid to measuring the risk premiums associated with these types of assets [10].

The object of an investment can be divided into two categories, namely assets with risk and assets without risk. This distinction alludes to whether the future performance cab be known beforehand. For example, stock prices and derivatives are risky assets, whereas bonds are risk-free assets. However, in order to maximize profit, an investor should have to ask himself the following question: What strategy is the best to maximize this investment?". In other words, "How many risky and risk-free assets would it take to maximize this investment?". Robert C. Merton solved this problem by determining the optimal fraction of risky and risk-free assets that maximizes the portfolio [6].

We pursue our objectives in the next three sections. Section 2 discusses the preliminaries, including the price dynamics of an asset that loses value as a regressive asset for which an acceptable definition is given. As well, the underlying equation

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receives due attention. In section 3, we present a model for the market and we determine the risk premium for a regressive asset while making some important observations about it. In the last section, we proceed with the dynamic programming of our model, determine the Merton fraction and make some remarks. This last section concludes with a numerical simulation showing the contribution of the optimal strategy on the value of the portfolio.

## 2. Preliminary

**Definition 2.1** (Brownian Motion)**.** A one dimensional Brownian Motion is an $\mathbb{R}$-valued process $B := (B_t)_{t\in[0,T]}$ such that:

($i$). $B_0 = 0$ $\mathbb{P}$-a.s

($ii$). $B$ has independent increments, i.e for every $0 \leq t < s \leq u < v$, the random variable $B_v - B_u$ and $B_t - B_s$ are independent;

($iii$). $B$ has stationary normally distributed increments with

$$B_t - B_s \sim \mathcal{N}(0, t-s), \quad \forall t > s \geq 0$$

($iv$). $B$ has $\mathbb{P}$-a.s continuous paths.

**Definition 2.2** (Itô Process)**.** Let $(B_t)_{t\in[0,T]}$ be a Brownian Motion defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and $\mathcal{F}^B$ its natural filtration. We call an Itô process any random process $(X_t)_{t\in[0,T]}$ of the form

$$X_t = X_0 + \int_0^t \mu_s ds + \int_0^t \sigma_s dB_s, \quad \forall t \in [0,T] \tag{2.1}$$

where $X_0$ is $\mathcal{F}_0^B$-adapted, the process $\left(\int_0^t \sigma_s dB_s\right)_{t\in[0,T]}$ is an Itô integral and the process $(\mu_t)_{t\in[0,T]}$ is $\mathcal{F}^B$-adapted process such that $\mathbb{E}\left[\int_0^T |\mu_s| ds\right] < \infty$.

**Corollary 2.3** (Itô product rule)**.** *Let $X$ be an Itô process given by (2.2) and $Y := (Y_t)_{t\in[0,T]}$ be another Itô process defined as following*

$$Y_t = Y_0 + \int_0^t a_s ds + \int_0^t \xi dB_s, \quad t \in [0,T]. \tag{2.2}$$

*Then, for every $t \in [0,T]$, it holds :*

$$X_tY_t = X_0Y_0 + \int_0^t X_s dY_s + \int_0^t Y_s dX_s + \langle X, Y\rangle_t. \tag{2.3}$$

**Definition 2.4** (Itô formula)**.** Let $(X_t)_{t\in[0,T]}$ denote an Itô process and $F : \mathbb{R}_+ \to \mathbb{R}$ be a bounded $\mathcal{C}^{1,2}$. Then it holds:

$$\begin{aligned} F(t, X_t) &= F(0, X_0) + \int_0^t \frac{\partial}{\partial t} F(s, X_s) ds + \int_0^t \frac{\partial}{\partial x} F(s, X_s) dX_s \\ &\quad + \frac{1}{2}\int_0^t \frac{\partial^2}{\partial x^2} F(s, X_s) d\langle X\rangle_s \end{aligned} \tag{2.4}$$

where $\langle X\rangle_s = \int_0^s \sigma_k^2 dk$, $\forall s < t$.

**Definition 2.5.** Let $(S_t)_{t\in[0,T]}$ be a stochastic process that describes the price of an asset. We will call regressive asset an asset for which the price has the following trend

$$\forall t_1, t_2 \in [0,T], t_1 < t_2, S_{t_1} \geq S_{t_2} \quad \text{(a.e)}, \tag{2.5}$$

which is a monotone decreasing process.

**Definition 2.6.** Let the following linear equation stochastic differential equation

$$\begin{cases} dS_t = -\mu S_t dt + \sigma S_t dW_t \\ \qquad S_0 \in \mathbb{R} \end{cases} \tag{2.6}$$

where : $\mu$ is the drift coefficient and $\sigma$ the diffusion coefficient which are strictly positive, $(W_t)_{t\in[0,T]}$ a Brownian Motion.

In the usual Black-Scholes model, $\mu \in \mathbb{R}$, which is different from our stochastic differential equation (2.6) where we fix $\mu$ to be strictly positive, and multiply it by the negative sign.

**Proposition 2.7.** *There exist only one solution $(S_t)_{t\in[0,T]}$ for equation (2.6) of the form*

$$S_t = S_0 \exp\left\{-\left(\mu + \frac{\sigma^2}{2}\right)t + \sigma W_t\right\}; \quad \forall t \in [0,T]. \tag{2.7}$$

*Proof.* By extending the Cauchy-Lipschitz theorem for SDEs we have the following. Let $f, g : \mathbb{R} \to \mathbb{R}$, the functions defined by $f(x) = -\mu x$ and $g(x) = \sigma x, \forall x \in \mathbb{R}$. Then, there is a constant $C > 0$, such that : $|f(x) - f(y)| + |g(x) - g(y)| \leq C.|x-y| \iff |\mu x - \mu y| + |\sigma x - \sigma y| = \mu^1|x-y| + \sigma|x-y| = (\mu+\sigma)|x-y| \leq C.|x-y|$ and $|f(x)| - |g(x)| \leq C(1+|x|) \iff |\mu x| - |\sigma x| = \mu|x| - \sigma|x| = (\mu-\sigma)|x| \leq (\mu+\sigma)(1+|x|)$ (We can take for example $C = \mu + \sigma$).

Now, taking $Y_t = -\left(\mu + \frac{\sigma^2}{2}\right)t + \sigma W_t$, we get $F(t, S_t) = S_t = S_0 \exp Y_t$ such that

$$\frac{\partial}{\partial t}F(t,S_t) = -\left(\mu + \frac{\sigma^2}{2}\right)S_t; \quad \frac{\partial}{\partial x}F(t,S_t) = \sigma S_t \quad \text{and} \quad \frac{\partial^2}{\partial x^2}F(t,S_t) = \sigma^2 S_t$$

where $x := W_t$. By applying the Itô's formula to the equation (2.7) we get

$$dS_t = -\mu S_t dt + \sigma S_t dW_t, \quad \forall t \in [0,T]. \tag{2.8}$$

□

In the following, we will use $S_t^0 := (S_t^0)_{t\in[0,T]}$ to describe the price of a risk-free asset.

[1] Since $\mu$ and $\sigma$ are strictly positives by definition.

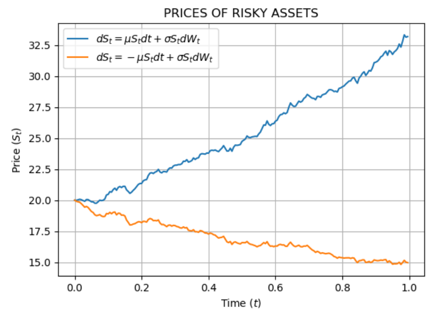


FIGURE 1. Dynamics of risky assets. With $T = 1$, $dt = 0.01$, $\sigma = 0.4$, $\mu = 0.8$ and $S_0 = 20$.

## 3. Modeling of the Market

In this section, $P$ and $Q$ are respectively the historical probability and the risk-neutral probability, $(\mathcal{F}_t)_{t\in[0,T]}$ is the filtration and $(B_t)_{t\in[0,T]}$ represents a Brownian Motion adapted to this filtration. $\mathcal{F}_t$ will represent, $\forall t \in [0,T]$, the price information $S_t$ available on the date $t$, such as

$$\mathcal{F}_t = \sigma(S_u : u \leq t), \quad \forall u, t \in [0,T]. \tag{3.1}$$

**Definition 3.1.** Let $S_t$ and $S_t^0$ respectively represent the prices of assets with and without risk at $t \in [0,T]$. We define the discounted price of the risky asset by

$$\tilde{S}_t = \frac{S_t}{S_t^0}, \quad \forall t \in [0,T]. \tag{3.2}$$

**Definition 3.2.** Let $(V_t(\pi_t))_{t\in[0,T]}$ be the value of the portofolio for a strategy $\pi_t = [\alpha_t, \beta_t]$ $\forall t \in [0,T]$. Then, the self-financing condition is given by

$$dV_t(\pi_t) = \beta_t dS_t^0 + \alpha_t dS_t, \quad \forall t \in [0,T]. \tag{3.3}$$

**Definition 3.3.** Let $(V_t(\pi_t))_{t\in[0,T]}$ be the value of the portofolio for a strategy $\pi_t := (\pi_t)_{t\in[0,T]}$. We define the present value of the portofolio for this same strategy by

$$\tilde{V}_t(\pi_t) = \frac{V_t(\pi_t)}{S_t^0}, \quad \forall t \in [0,T]. \tag{3.4}$$

We have the following:

**Proposition 3.4.** *Let $\tilde{V}_t(\pi) := (\tilde{V}_t(\pi_t))_{t\in[0,T]}$ be the present value of the portofolio, $\alpha_t = (\alpha_t)_{t\in[0,T]}$ the quantity of risk assets and $\tilde{S}_t := (\tilde{S}_t)_{t\in[0,T]}$ the discounted price of risky asset. Then, the dynamic of the present value of the portfolio is given by*

$$d\tilde{V}(\pi_t) = \alpha_t d\tilde{S}_t, \quad \forall t \in [0,T]. \tag{3.5}$$

*Proof.* By applying the Itô product rule to the present value of the portfolio for the $\pi := \pi_t$ strategy, we obtain

$$\tilde{V}_t(\pi) = \tilde{V}_0(\pi) - \int_0^t \frac{V_u(\pi)dS_u^0}{(S_u^0)^2} + \int_0^t \frac{dV_u(\pi)}{S_u^0}, \tag{3.6}$$

by integrating equation (3.3) in equation (3.6), we get

$$\begin{aligned}\tilde{V}_t(\pi) &= \tilde{V}_0(\pi) - \int_0^t \frac{V_u(\pi)dS_u^0}{(S_u^0)^2} + \int_0^t \frac{\beta_u dS_u^0 + \alpha_u dS_u}{S_u^0} \\ &= \tilde{V}_0(\pi) + \int_0^t \alpha_u \left[\frac{-S_u dS_u^0}{(S_u^0)^2} + \frac{dS_u}{S_u^0}\right]\end{aligned} \tag{3.7}$$

By applying the Itô product rule on $\tilde{S}_t$ we get

$$\begin{aligned}\tilde{S}_t &= \tilde{S}_0 + \int_0^t \frac{dS_u}{S_u^0} - \int_0^t \frac{S_t}{(S_t^0)^2} dS_u^0 \\ &= \tilde{S}_0 + \int_0^t \left[\frac{dS_u}{S_u^0} - \frac{S_u dS_u^0}{(S_u^0)^2}\right] \implies d\tilde{S}_t = \frac{dS_t}{S_t^0} - \frac{S_t dS_t^0}{(S_t^0)^2} \quad \forall t \in [0,T].\end{aligned} \tag{3.8}$$

Then, setting $S_0^0 = 1$, we arrive at

$$\tilde{V}_t(\pi) = \tilde{V}_0(\pi) + \int_0^t \alpha_u d\tilde{S}_u, \quad \forall t \in [0,T]. \tag{3.9}$$

□

**Definition 3.5** (Risk Premium)**.** The risk premium is the additional return on investment offered by an asset compared to a risk-free investment.

This additional return remunerates the investor for his greater risk-taking. The value of the risk premium depends on variations in the risk level of an asset or a stock index. In general, riskier assets have a higher risk premium in theory. We can see, according to this definition of "Café de la Bourse," that the more our assets are risky, the higher the risk premium is.

**Theorem 3.6** (Girsanov theorem)**.** *Let $(\Theta_t)_{t\in[0,T]}$ be a stochastic process such that $(L_t)_{t\in[0,T]}$ defined by*

$$L_t = \exp\left\{-\int_0^t \Theta_s dB_s - \frac{1}{2}\int_0^t \Theta_s^2 ds\right\}, \quad \forall t \in [0,T]$$

*is a martingale. Then, under the probability measure $Q$ which is absolutely continuous with respect to $P$ ($dQ = L_t dP$), the process*

$$\left(B_t + \int_0^t \Theta_s ds\right)_{t\in[0,T]}$$

*is a Brownian Motion, where*

$$\mathbb{E}^p\left[\exp\left\{-\int_0^t \Theta_s dB_s - \frac{1}{2}\int_0^t \Theta_s^2 ds\right\}\right] < \infty$$

**Proposition 3.7.** *Under the probability $Q$, we define the Brownian Motion $(W_t)_{t\in[0,T]}$ by*

$$W_t = \lambda t - B_t, \quad t \in [0,T], \tag{3.10}$$

*such that*

$$\lambda = \frac{r+\mu}{\sigma} \tag{3.11}$$

*is the risk premium relative to our regressive assets, where $r > 0$ is the interest rate of the risk-free asset and*

$$d\tilde{S}_t = \sigma \tilde{S}_t dW_t, \quad \forall t \in [0,T]. \tag{3.12}$$

*Proof.* Firstly, let $(L_t)_{t\in[0,T]}$ be defined by

$$\begin{aligned} L_t &= \exp\left\{-\int_0^t \frac{r+\mu}{\sigma} dB_s - \frac{1}{2}\int_0^t \left(\frac{r+\mu}{\sigma}\right)^2 ds\right\} \\ &= \exp\left\{-\frac{r+\mu}{\sigma}(B_t - B_0) - \frac{1}{2}\left(\frac{r+\mu}{\sigma}\right)^2 t\right\}, \end{aligned}$$

by taking $\lambda = \dfrac{r+\mu}{\sigma}$, we get

$$L_t = \exp\left\{-\lambda B_t - \frac{1}{2}\lambda^2 t\right\}. \tag{3.13}$$

We now show that $L_t$ is a martingale. Indeed, $\forall s,t \in [0,T]$ such that $s \leq t$,

$$\begin{aligned} \mathbb{E}[L_t|\mathcal{F}_s] &= \mathbb{E}\left[\exp\left\{-\lambda B_t - \frac{1}{2}\lambda^2 t\right\}|\mathcal{F}_s\right] \\ &= e^{-\frac{\lambda^2 t}{2}}\,\mathbb{E}\left[\exp\left\{-\lambda(B_t - B_s + B_s)\right\}|\mathcal{F}_s\right] \\ &= e^{-\frac{\lambda^2 t}{2}} e^{-\lambda B_s}\mathbb{E}\left[\exp\left\{-\lambda(B_t - B_s)\right\}|\mathcal{F}_s\right]. \end{aligned}$$

Since $B_t - B_s \sim \mathcal{N}(0, t-s)$,

$$\begin{aligned} \mathbb{E}[L_t|\mathcal{F}_s] &= \exp\left\{-\frac{\lambda^2 t}{2} - \lambda B_s + \frac{\lambda^2}{2}(t-s)\right\} \\ &= \exp\left\{-\left(\frac{r+\mu}{\sigma}\right) B_s - \frac{1}{2}\left(\frac{r+\mu}{\sigma}\right)^2 s\right\}, \end{aligned}$$

hence, $(L_t)_{t\in[0,T]}$ is a martingale. According to Girsanov's theorem, there exists a process $(W_t)_{t\in[0,T]}$ such that

$$W_t = B_t + \int_0^t \frac{r+\mu}{\sigma} ds, \quad t \in [0,T],$$

is a Brownian motion. Since in equation (3.10) we have the symmetry of $B_t$ which has the same distribution as $B_t$ ($B_t \overset{Law}{=} -B_t$). Then

$$W_t := \int_0^t \frac{r+\mu}{\sigma} ds - B_t = \lambda t - B_t, \quad \forall t \in [0,T],$$

is a Brownian motion.

Secondly,

$$W_t = \lambda t - B_t \Longleftrightarrow \sigma W_t = (r+\mu)t - \sigma B_t \Longleftrightarrow \sigma dW_t = (\mu + r)dt - \sigma dB_t.$$

On the other hand, we know that $\tilde{S}_t = \dfrac{S_t}{S_t^0}$ and therefore, by applying the Itô's product rule, we have

$$\tilde{S}_t = \tilde{S}_0 + \int_0^t \frac{1}{S_u^0}\left[\frac{-S_u dS_u^0}{S_u^0} + dS_u\right] = \tilde{S}_0 + \int_0^t \tilde{S}_u[-\sigma dB_u + (\mu + r)du],$$

hence,

$$d\tilde{S}_t = \tilde{S}_t[(\mu + r)du - \sigma dB_u] = \sigma \tilde{S}_t dW_t, \quad \forall t \in [0,T]. \tag{3.14}$$

□

*Remark* 3.8. Contrary to the usual Black-Scholes model, this risk premium cannot, $\forall t \in [0,T]$, be zero. In the case where $\mu = r$ we observe that $\lambda = \dfrac{2r}{\sigma}$. This confirms the fact that, $\forall t \in [0,T]$, the risk premium for a regressive asset is higer.

## 4. Dynamic programming of a regressive asset and numerical simulation

In this section, we are interested in the dynamic programming of the regressive asset and in applying the optimal stochastic control to it. Based on the exponential structure of the equation 4.9, we selected the logarithmic function for the utility function.

Let the following stochastic differential equation represent the dynamics of a one-dimensional controlled state $(X_t)_{t\in[0,T]}$

$$\begin{cases} dX_t = a(t, X_t, \pi_t)dt + b(t, X_t, \pi_t)dW_t \\ \qquad X_0 = x_0 \in \mathbb{R} \end{cases} \tag{4.1}$$

with $a, b : [0,T] \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. We will assume that the control (**Markovian-control**) $\pi_t := (\pi_t)_{t\in[0,T]}$ is $\mathcal{F}$-adapted and has value in $\mathbb{R}$ $\forall t \in [0,T]$.

**Definition 4.1.** When we restrict $\pi_t$ to $\pi(t, X_t)$ $\forall t \in [0,T]$, $\tilde{\pi}$ is called decision rule.

**Definition 4.2** (Performance criterion or Reward function)**.** We define the performance criterion by

$$\mathcal{J}(x_0; \pi) = \mathbb{E}\left[\int_0^T F(s, X_s^\pi, \pi_s)ds + \Phi\left(X_T^\pi\right)\right] \tag{4.2}$$

where $F$ is the running reward $F : [0,T] \times \mathbb{R} \times \mathcal{K} \subset \mathbb{R} \to \mathbb{R}$ and $\Phi$ is the terminal reward $\Phi : \mathbb{R} \to \mathbb{R}$.

**Definition 4.3.** $(\pi_t)_{t\in[0,T]}$ with $\pi_t = \tilde{\pi}(t, X_t)$ is said to be admissible when:

(*i*). $\tilde{\pi}(t, X_t) \in \mathcal{K}, \quad \forall (t,x) \in [0,T] \times \mathbb{R}.$

$(ii)$. For every initial value $(t, x)$, the stochastic differential equation

$$\begin{cases} dX_s = a(s, X_s, \tilde{\pi}(s, X_s))ds + b(s, X_s, \tilde{\pi}(s, X_s))dW_s \\ X_t = x \in \mathbb{R} \end{cases}, \quad \forall s \in [s, T] \tag{4.3}$$

has a unique solution, for which the performance criterion $\mathcal{J}_0$ in equation (4.2) is

$$\mathbb{E}\left[\int_0^T |F(s, X_s^\pi, \tilde{\pi}(s, X_s))|ds + |\Phi\left(X_T^\pi\right)|\right] < \infty \tag{4.4}$$

**Definition 4.4.** We call the value function of the control problem

$$\mathcal{V}_0(x_0) = \sup_{\pi \in \mathcal{A}} \mathcal{J}_0(x_0; \pi) \tag{4.5}$$

where $\mathcal{A}$ is the set of admissible controls.

*Remark* 4.5. If there exist $\pi^* \in \mathcal{A}$ control such as

$$\mathcal{J}_0(x_0; \pi^*) = \mathcal{V}_0(x_0),$$

then $\pi^*$ is called optimal control and we have the following equality

$$X^{*\pi} = X^{\pi^*}$$

The basic notions of optimal stochastic control having been stated, we can now determine the control on our regressive asset.

Let the wealth invested or the value of the portfolio be defined by $V := (V_t)_{t \in [0,T]}$. Let the condition of self-financing be defined by 3.3. In the following, we will use $\pi_t := (\pi_t)_{t \in [0,T]}$ for the investment strategy $\forall t \in [0, T]$, $\alpha_t = (\alpha_t)_{t \in [0,T]}$ the number of risky-assets and $\beta_t := (\beta_t)_{t \in [0,T]}$ the number of assets without risk.

The fraction of the portfolio value (fraction of wealth) invested in risky assets (stocks) and risk-free assets (bounds) are respectively:

$$\pi_t = \frac{\alpha_t S_t}{V_t} \quad \text{and } 1 - \pi_t = \frac{\beta_t S_t^0}{V_t} \tag{4.6}$$

Consider the following system of equations

$$\begin{cases} dS_t^0 = rS_t^0 dt \\ dS_t = -\mu S_t dt + \sigma S_t dW_t \end{cases}, \quad \forall t \in [0, T] \tag{4.7}$$

modelling the dynamics of assets without and with risks. By taking the expressions of $\alpha_t$ and $\beta_t$ from the equation (4.6) and replacing then in (3.3) we get

$$dV_t^\pi = V_t^\pi((r - (r + \mu)\pi_t)dt + \sigma\pi_t dW_t), \quad \forall t \in [0, T]. \tag{4.8}$$

**Proposition 4.6.** *Let the stochastic differential equation (4.8) define the self-financing dynamic. Then, the solution of equation (4.8) is of the form*

$$V_t^\pi = V_t(\pi) = V_0 \exp\left\{\int_0^t \left(r - (r + \mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2\right) ds + \int_0^t \sigma\pi_s dW_s\right\} \tag{4.9}$$

*Proof.* Let $Y_t = \int_0^t (r-(r+\mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2)ds + \int_0^t \sigma\pi_s dW_s$. We get[2] $V_t(\pi) = V_0 e^{Y_t}$ such that

$$\frac{\partial}{\partial t}V_t(\pi) = (r-(r+\mu)\pi_t - \frac{1}{2}(\sigma\pi_t)^2)V_t(\pi); \quad \frac{\partial}{\partial x}V_t(\pi) = \sigma\pi_t V_t(\pi)$$

$$\text{and} \quad \frac{\partial^2}{\partial x^2}V_t(\pi) = (\sigma\pi_t)^2 V_t(\pi)$$

with

$$\langle W\rangle_t = \left\langle B_. + \int_0^{.} \frac{r+\mu}{\sigma} d. \right\rangle_t = \langle B_.\rangle_t = t$$

where $x := W_t$. By applying The Itô's formula to the equation (4.9) we get

$$\begin{aligned} dV_t^\pi &= (r-(r+\mu)\pi_t - \frac{1}{2}(\sigma\pi_t)^2)V_t^\pi dt + \sigma\pi_t V_t^\pi dW_t + \frac{1}{2}(\sigma\pi_t)^2 V_t^\pi dt \\ &= V_t^\pi((r-(r+\mu)\pi_t)dt + \sigma\pi_t dW_t), \quad \forall t \in [0,T]. \end{aligned}$$

□

Let the performance criteria be defined by

$$\mathcal{J}(x_0;\pi) = \mathbb{E}\left[\mathrm{U}(V_T^\pi)\right] \to \max_\pi \tag{4.10}$$

We are interested in the expected terminal utility instead of running wealth. For simplicity we use the log utility function thanks to the relationship between logarithmic and exponential functions. Then, our performance criterion defined by equation (4.10) becomes

$$\mathcal{J}(x_0;\pi) = \mathbb{E}\left[\log V_T^\pi\right] \tag{4.11}$$

knowing that

$$\log V_T^\pi = \log V_0^\pi + \int_0^T \left(r-(r+\mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2\right)ds + \int_0^T \sigma\pi_s dW_s$$

this implies that

$$\mathbb{E}\left[\log V_T^\pi\right] = \mathbb{E}\left[\log V_0^\pi\right] + \mathbb{E}\left[\int_0^T \left(r-(r+\mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2\right)ds\right] + \mathbb{E}\left[\int_0^T \sigma\pi_s dW_s\right]$$

knowing that, by the martingality of the stochastic integral with respect to the Brownian motion, we have

$$\mathbb{E}\left[\int_0^T \sigma\pi_s dW_s\right] = 0$$

$$\Longrightarrow \mathbb{E}\left[\log V_T^\pi\right] = \mathbb{E}\left[\log V_0^\pi\right] + \mathbb{E}\left[\int_0^T \left(r-(r+\mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2\right)ds\right]$$

$\mathbb{E}\left[\log V_0^\pi\right]$ being a constant we will maximize the second expectation. Let's take

$$g(\pi_s) = r-(r+\mu)\pi_s - \frac{1}{2}(\sigma\pi_s)^2$$

[2] $V_0 := V_0(\pi)$.

and we get

$$\pi^* = -\frac{(r+\mu)}{\sigma^2} \tag{4.12}$$

*Remark* 4.7. Contrary to the usual Black-Scholes model, the Merton fraction never vanishes for all $t \in [0, T]$ and we have that when $r = \mu$, $\pi^* = -\frac{2r}{\sigma}$. In addition, the optimal strategy cannot be pure bond (buy only risk-free assets) since $(r+\mu) \neq 0$, $t \in [0, T]$ or pure stock (buy only risky assets) since $(r+\mu) \neq -\sigma^3$ for all $t \in [0, T]$ when the asset price is regressive.

Then, we can conclude by making the following proposition.

**Proposition 4.8.** *Let $\lambda = (\lambda_t)_{t\in[0,T]}$ define the risk premium of a risky asset and $\sigma = (\sigma_t)_{t\in[0,T]}$ its volatility. Then, the Merton fraction is given by*

$$\pi^* = -\frac{\lambda}{\sigma}. \tag{4.13}$$

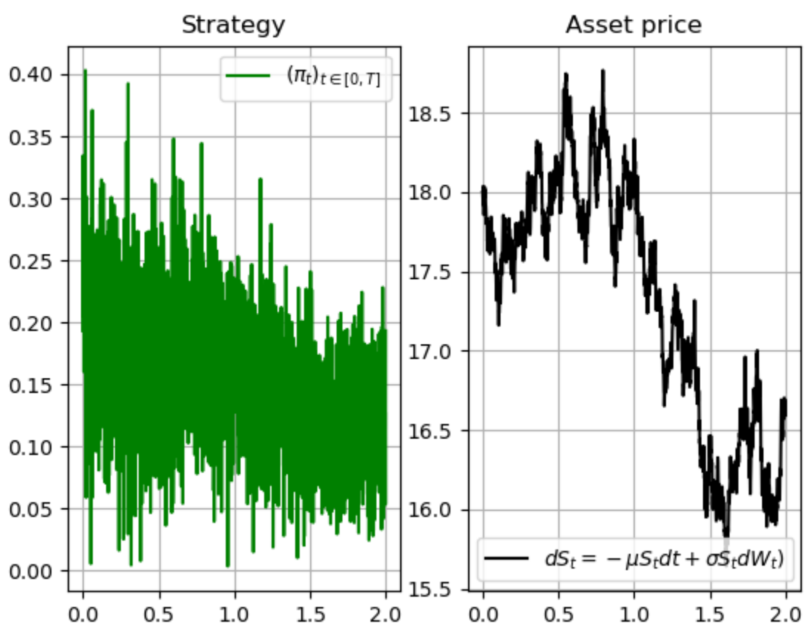


FIGURE 2. Dynamics of risky assets with $T = 2$, $\sigma = 0.74$, $\mu = 0.13$, $r = 0.134$ and $S_0 = 18$.

[3]Since $r, \mu > 0$.

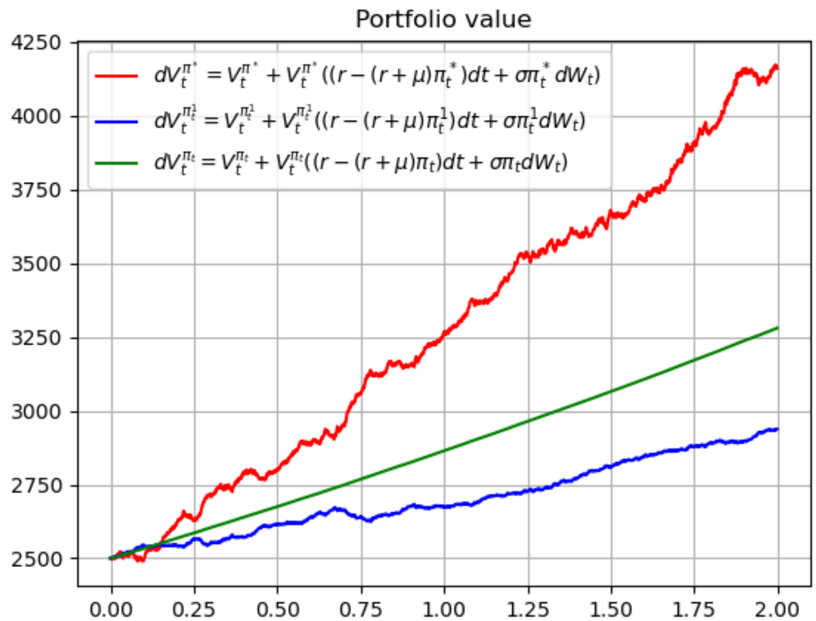


FIGURE 3. Dynamics of the portfolio $T = 2$, $\sigma = 0.74$, $\mu = 0.13$, $r = 0.134$ and $V_0 = 2500$.

We observe that the value of the portfolio increases when the strategies are respectively $\pi^*$ and $\pi_t^1 = \dfrac{S_t \alpha_t}{V_t}$, $\forall t \in [0, T]$. However, when the strategy is $\pi = \dfrac{\mu - r}{\sigma^2}$, as classically presented by several authors, we notice that the value of the portfolio has a small increasing trend.

We can understand that the optimal strategies depend on the nature of the price and therefore on the nature of its parameters, in order to expect maximum portfolio value. Thus, we also notice that the negative values of the strategy $\pi$ mean "sell the risky assets".

## 5. Conclusion

In this paper we have shown that the Black-Scholes model, although critical, could be a good interpreter in the world of finance with the above determination of the risk premium being a brief example. We have shown that the more an asset loses value, the greater the return on investment. By doing dynamic stochastic programming on this regressive asset we were able to determine the Merton fraction, and state a proposition on the relationship between return on investment and optimal strategy. This has been verified by a numerical simulation.

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Jean Auguste MUHAU: Faculté des Sciences Technologiques, Université Nouveaux Horizons, 2465 Route Kasapa, Lubumbashi, Democratic Republic of Congo

*Email address*: `auguste.muhau@unhorizons.org,augustemuhau@gmail.com`

Julius N. Esunge: Department of Mathematics, University of Mary Washington, Fredericksburg, VA, 22401, USA

*Email address*: `jesunge@umw.edu`